Abstract

The flow-turning effect in rocket stability prediction has been the focus of controversy since it was first introduced several decades ago. Because it arose in a strictly one-dimensional context, its incorporation into three-dimensional motor stability computations requires further justification. A complete three-dimensional unsteady viscous analytical solution for the flow in a rocket chamber with a realistic mean flow field is used in this paper to establish the three-dimensional flow-turning stability integral and to justify its use in stability calculations. The key to a physical understanding of flow-turning is recognition of the role played by vorticity in acoustic/mean flow interactions. Crocco’s theorem clarifies the origin of unsteady vorticity in the presence of axial acoustic waves. The transfer of energy from acoustic pressure oscillations to vortical waves provides the mechanism by which gas particles produced in the combustion process acquire the axial motion of the acoustic wave. However, Culick’s original analysis is incomplete because it did not account for the effect of these processes on the unsteady normal flow field near the propellant surface. The corrected radial velocity fluctuation leads to an additional driving effect. Hence the net stabilizing influence is significantly smaller than predicted by the classical flow-turning model in typical burner geometries.

Introduction

Culick’s papers on combustion instability\(^1\)-\(^4\) published in the early 1970’s are the foundation for all stability prediction methods in current use.\(^5\)-\(^6\) An important feature of his one-dimensional formulation is the requirement that the unsteady gas flow must enter the chamber in a direction normal to the boundary. Hence, the equivalent of the multidimensional no-slip condition is imposed. This is not a requirement of the one-dimensional approach, which allows specification of any angle of flow injection into the chamber. A consequence of this boundary constraint is the appearance of a term in the motor stability prediction, the flow-turning effect, that represents energy lost as the incoming flow acquires the axial motion of the one-dimensional acoustic wave.

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A similar damping effect does not appear in the three-dimensional stability formulations.\textsuperscript{5-6} The absence of flow-turning in the standard multidimensional stability calculation is due to the failure to impose the no-slip condition. Culick’s assumption that the latter can be corrected by simply superimposing the one-dimensional result has been the focus of considerable discussion.

An important difficulty in resolving such questions is that the one-dimensional theory does not allow a thorough physical interpretation of the flow-turning effect because of the limitations imposed by the mathematical formulation. For example, it is not possible to accommodate features such as production and transport of vorticity although such flow effects are obviously introduced when the no-slip condition is invoked.

Earlier works suggested that the flow-turning loss is related to “viscous processes” in the vicinity of the burning zone.\textsuperscript{3-4,7} This implies that mechanical energy dissipation is involved. It is necessary that all such questions be fully resolved if a practical and dependable multidimensional combustion stability theory and prediction methodology are to be attained.

The goals of the work described in this paper are to:

- Clarify the physical origin of flow-turning
- Extend to a fully three-dimensional model
- Determine the dependence of flow-turning on viscous effects

The last is necessary since recent results suggest that viscous forces may not always play a crucial role.\textsuperscript{8} These goals are accomplished by means of a fully viscous multidimensional analytical solution for a realistic cylindrical chamber and mean flow configuration. The analytical approach allows a much more detailed physical interpretation of the phenomenon than is available in the purely numerical calculations used by others.

The analysis gives rise to a multidimensional stability correction that is identical to the classical one-dimensional flow-turning; Culick’s original “patching” procedure is thus validated. It is demonstrated that flow-turning is a consequence of the production of vorticity waves in the acoustics/mean flow interaction. The flow-turning stability integral can be collapsed to an integral over the chamber burning surfaces, making it applicable to any rocket chamber geometry, and simplifying its incorporation in prediction algorithms.

However, an important element not appearing in the one-dimensional formulation is a correction to the radial velocity fluctuation at the injection surface that accompanies the flow-turning process. Continuity requires this correction since the axial velocity fluctuation must vanish at the boundary. The result is an additional driving effect in the stability energy balance that has not been accounted for in previous prediction algorithms. In simple motor configurations (such as a full-length cylindrical grain) this correction exactly cancels the classical flow-turning loss. The net flow-turning effect can be either driving or damping depending on the combustion chamber geometry. The corrected stability algorithm predicts significantly less stable (or more unstable) behavior for many motor chamber designs.

### Analysis

An irrotational (acoustic) unsteady flow is not a sufficiently complete model of the combustion chamber motion. Such a model allows slip flow at the propellant surfaces. Acoustic boundary layer theory, which has been applied to this problem frequently in the past, may not be appropriate since the strong convective field in the chamber transports vorticity throughout the domain.

In this situation, it is not clear at the outset what role, if any, is played by viscosity since the radial convection tends to discourage formation of highly sheared regions of flow. It is also important to realize that the creation of vorticity does not require the presence of viscous stresses. Crocco’s theorem\textsuperscript{8} clarifies the origin of both the steady and unsteady vorticity in the gas particles entering the chamber from the burning zone.

### Formulation

Most of the standard assumptions used in combustion stability modeling are employed here. A standard set of dimensionless variables is employed. Velocities are made dimensionless with respect to the chamber sound speed, \( a_0 \), to emphasize the central role of compressibility in the oscillating field. Lengths are referenced to the chamber radius, \( R \), mainly for convenience in plotting radial distributions of results. The acoustic wavelength expressions then contain the chamber length-to-radius ratio since longitudinal oscillations will be of primary concern. Time is made dimensionless by dividing by the characteristic time represented by the ratio \( R/a_0 \).

Pressure is normalized by \( \gamma P_0 \) and other thermodynamic variables are nondimensionalized with respect to the respective chamber stagnation properties.

The momentum balance for the first-order unsteady viscous flow is

\[
\frac{\partial u^{(l)}}{\partial t} + \nabla p^{(l)} = -M_b \begin{pmatrix}
\nabla \left( u^{(l)} \cdot U \right) - u^{(l)} \times \nabla \times U \\
-U \times \nabla \times u^{(l)} \\
+ \frac{\delta^2}{M_b} \nabla \times \nabla \times u^{(l)}
\end{pmatrix}
\] (1)
where superscript (1) indicates that terms of first-order are retained. Conservation of mass requires that
\[ \frac{\partial p^{(1)}}{\partial t} + \nabla \cdot u^{(1)} = -M_b \mathbf{U} \cdot \nabla p^{(1)} \] (2)
Assuming sinusoidal oscillations, \( p^{(1)} = p' \exp(-ik_m t) \) and \( u^{(1)} = u' \exp(-ik_m t) \) where primes denote the complex amplitudes, the equation for the unsteady vorticity amplitude, \( \omega' = \nabla \times u' \), is found by taking the curl of Eq. (1) with the result
\[ ik_m \omega' = -M_b \nabla \times [u' \times \Omega + U \times \omega'] + \delta^2 \nabla \times \nabla \times \omega' \tag{3} \]
In the present application to a cylindrical geometry with burning at the sidewall and an axial acoustic wave, the vorticity vector has only an azimuthal component (as is also true in the corresponding steady flow\(^9\)).

The boundary condition that must be satisfied by the vorticity at the burning surface is found by evaluating the axial component of Eq. (1) at \( r = 1 \) and solving for terms involving the vorticity amplitude. One finds that
\[ \omega' + \frac{\delta^2}{M_b} \left( \frac{\partial \omega'}{\partial r} + \omega' \right) = -\frac{1}{M_b} \frac{\partial \omega'}{\partial z} + \frac{\partial u'}{\partial z} + \pi^2 zu' \tag{4} \]
at the burning surface. The no-slip condition has been applied. The last two terms represent the effects of radial velocity fluctuations at the burning surface. These are very small (of the order of the mean flow Mach number, \( M_b \)). Thus, one must represent the unsteady vorticity as a combination of irrotational and rotational parts. Put
\[ u' = \hat{u} + \bar{u} \tag{5} \]
where the circumflex (\( \hat{u} \)) indicates the acoustic (irrotational) part whereas the tilde (\( \bar{u} \)) identifies the rotational component.

For a three-dimensional cylindrical chamber, the axial acoustic motion is the simple plane wave solution
\[ \begin{align*}
    p' &= \cos(k_m z) \\
    \bar{u} &= \hat{\omega} e_z = i \sin(k_m z) e_z
\end{align*} \tag{6} \]
Use of this solution in the stability calculations does not produce a flow-turning contribution. This is a result of the failure of Eq. (7) to satisfy the no-slip condition at the burning surface. Therefore, it is necessary to investigate the rotational unsteady flow corrections to the acoustic motion before further progress can be made.

**Unsteady Vorticity**

In order to determine the correct form for the rotational unsteady velocity, \( \bar{u} \), it is first necessary to solve for the unsteady vorticity from Eq. (3) subject to the boundary condition of Eq. (4), which, as has already been noted is the equivalent of the no-slip condition. Once \( \omega' \) is known, the momentum and continuity balances, Eqs. (1) and (2) can be solved for the rotational velocity vector. For the assumed cylindrical geometry, Eq. (3) is
\[ ik_m \omega' - M_b \left( U_r \frac{\partial \omega'}{\partial r} + U_z \frac{\partial \omega'}{\partial z} - \frac{U_z}{r} \omega' \right) = \]
\[ -\delta^2 \left( \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} + \frac{\partial^2 \omega'}{\partial z^2} \right) + M_b \left( \Omega \left( ik_m p' - \frac{u^{(1)}}{r} \right) + \right. \]
\[ \left. + u^{(1)} \frac{\partial Q}{\partial r} + w^{(1)} \frac{\partial Q}{\partial z} \right) \tag{8} \]
This appears to be a quite difficult equation to solve due primarily to the variable coefficients representing the steady velocity components and vorticity, \( \Omega \). The latter are given by the Culick mean flow model as
\[ \begin{align*}
    U_r &= -\sin\left(\frac{1}{2} \pi r^2\right) \\
    U_z &= \pi z \cos\left(\frac{1}{2} \pi r^2\right) \\
    \Omega &= \pi^2 z r \sin\left(\frac{1}{2} \pi r^2\right)
\end{align*} \tag{9} \]
A simple perturbation approach suggests that the vorticity is of the order of the mean flow Mach number, \( M_b \). This is the origin of the commonly stated opinion that the vorticity effects in combustion instability are of “higher-order” and presumably negligible.
The complete solution to Eq. (8) is most readily constructed by recognizing that there are two types of oscillatory behavior involved. The first type is related to the acoustic interaction terms on the right hand side. These vary slowly with position like the product of the mean vorticity with the acoustic pressure or velocity. The second type of vortical motion involves rapid variations in the radial direction due to the production of vorticity waves as will be demonstrated shortly. Hence, it is useful to separate the two types of solutions by writing

\[ \omega' = \tilde{\omega} + \tilde{\omega} \]  

where the notation is consistent with that introduced earlier. This allows Eq. (8) to be separated into two independent parts:

\[ \begin{align*}
    ik_m \hat{\omega} &= M_b \left[ ik_m \rho' \Omega + \dot{\omega} \frac{\partial \Omega}{\partial \zeta} \right] \\
    \frac{\partial \hat{\omega}}{\partial r} + U_r \frac{\partial \hat{\omega}}{\partial \zeta} - \frac{\omega}{r} - i \frac{k_m}{M_b U_r} \hat{\omega} &= \\
    = \frac{\delta^2}{M_b U_r} \left( \frac{\partial^2 \hat{\omega}}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{\omega}}{\partial r} - \frac{\hat{\omega}}{r^2} + \frac{\partial^2 \hat{\omega}}{\partial \zeta^2} \right) - \frac{\partial \Omega}{\partial \zeta} \frac{\partial \hat{\omega}}{\partial U_r} \tag{14}
\end{align*} \]

Terms involving the radial velocity fluctuation in Eq. (8) have been dropped since they introduce terms of only second-order in \( M_b \). The rotational part, Eq. (14), has been written in wave equation form by dividing through by the radial steady velocity at the surface, \( M_b U_r \).

The solution of Eq. (13) is straightforward. For a cylindrical chamber the result is

\[ \hat{\omega} = \frac{M_b}{k_m} \pi^2 r \sin \left( \frac{1}{2} \pi r^2 \right) \left( k_m z \cos \left( k_m z \right) + \sin \left( k_m z \right) \right) \]  

for the small part of the vorticity due to interaction of the mean flow with the irrotational unsteady field.

Solution of Eq. (14) is more involved. It is useful to note first that Eq. (14) is a perturbed first-order wave equation. This suggests application of the ansatz

\[ \begin{align*}
    \hat{\omega} &= \zeta \exp(i \psi(r)) \sin \left( \frac{1}{2} \pi r^2 \right) \\
    \hat{w} &= W \exp(i \psi(r)) \sin \left( \frac{1}{2} \pi r^2 \right) \tag{16}
\end{align*} \]

where \( \zeta(r) \) and \( W(r) \) are functions of radial position to be determined. The sinusoidal factor containing the axial position dependence is based on the \( \chi(r, z) \) factor of Reference 8, which was expressed in series form as

\[ \chi(r, z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j + 1)!} (k_m z)^{2j} \sin^{2j} \left( \frac{1}{2} \pi r^2 \right) \]  

This series can be written in closed form \( \tag{18} \) as

\[ \chi(r, z) = \left[ \sin \left( \frac{1}{2} \pi r^2 \right) \right]^{-1} \sin \left( k_m z \sin \left( \frac{1}{2} \pi r^2 \right) \right) \]  

showing the origin of the factor used in the trial solutions of Eqs. (16) and (17). The exponential argument

\[ \psi(r) = -\frac{k_m}{\pi M_b} \ln \tan \left( \frac{1}{2} \pi r^2 \right) \]  

is also based on the findings of Reference 8. This is plotted in Figure 1 and shows that the vortical motion is characterized by spatial oscillations of increasing frequency as radial position, \( y = 1 - r \) increases.

![Fig. 1. Effect of Strouhal Number, S, on \( \psi \).](image)

Inserting Eqs. (16) and (17) into (14) and simplifying yields a first-order differential equation for \( \zeta(r) \):

\[ \begin{align*}
    \frac{\partial \zeta}{\partial r} + i \frac{\dot{\zeta}}{r} - i \frac{k_m}{M_b U_r} \zeta &= \\
    + \frac{\delta^2}{M_b U_r} \left( \frac{\partial \psi}{\partial r} \right)^2 + \frac{\partial \Omega}{\partial \zeta} \frac{\partial \psi}{\partial U_r} &= 0 \tag{21}
\end{align*} \]

where only the significant terms have been retained. For example, the only part of the viscous force that affects the solution is that part arising from the second derivative of \( \psi \) with respect to \( r \) in Eq. (14). When this second derivative is evaluated, it happens that only the part involving the square of the derivative of \( \psi \) gives rise to a significant influence on the solution. This is because

\[ \frac{d^2 \psi}{dr^2} = \frac{k_m}{M_b U_r} \]  

is inversely proportional to the (small) mean flow Mach number. Hence the square of this term dominates the viscous force as indicated in Eq. (21).

The axial momentum equation \( (z\text{-component of Eq. (1)}) \) can be used to express function \( W(r) \) (which is required to complete the solution of Eq. (21), in terms of the vorticity.)
One finds
\[ ik_m \vec{w} = M_b \left[ \frac{\partial}{\partial z} (U_z w') - U_z \omega' \right] + \delta^2 \frac{1}{r} \frac{\partial}{\partial r} (r \omega') \] (23)
where the radial velocity terms have been omitted as before since they would introduce corrections of second-order in the mean Mach number. Using Eqs. (16) and (17), the axial vortical velocity function is
\[ W = \left( \frac{M_b}{k_m} U_r - i \frac{\delta^2}{M_b U_r} \right) \zeta \] (24)
to good approximation. Substituting this result yields the first-order differential equation for the vorticity function
\[ \frac{d\zeta}{dr} + \left( \frac{\zeta}{U_r} - 1 - \pi^2 \frac{\zeta}{S^2 U_r} - i \frac{\pi^2}{S} r^2 U_r \right) \zeta = 0 \] (25)
where the first term in the brackets is the damping effect due to viscosity. The solution is controlled by two dimensionless parameters, the Strouhal number,
\[ S = \frac{k_m}{M_b} \] (26)
and the viscosity parameter
\[ \xi = \frac{k_m^2 \delta^2}{M_b^2} = \frac{S^2 \delta^2}{M_b} \] (27)
which is a combination of the acoustic Reynolds number, Strouhal number, and mean flow Mach number. Table 1 shows values of these key parameters for typical rocket motor configurations.

Equation (25) is readily solved by simple quadrature. One finds
\[ \zeta = r C \exp(\phi) \] (28)
where the complex coefficient \( C \) must be found from the boundary condition expressed Eq. (4). The complex argument \( \phi(r) \) arises in the integration of the right side of Eq. (25). Part of the integration cannot be carried out in finite form, since it involves the integral
\[ I(r) = \int \frac{x}{\sin(x)} \, dx \] (29)
which must be represented by the infinite series
\[ I(r) = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} - 1}{(1 + 2k)(2k)!} B_{2k} x^{1+2k} = \]
\[ = x + \frac{x^3}{18} + \frac{7x^5}{1800} + \frac{31x^7}{105840} + \frac{127x^9}{5443200} + \cdots \] (30)
The transformation
\[ x = \frac{1}{2} \pi r^2 \] (31)
has been used here for convenience. Functions \( B_{2k} \) are the Bernoulli numbers \( B_2 = 1/6, \ B_4 = -1/30, \text{ etc.} \).

The remainder of the integration is straightforward, and one finds for the real and imaginary parts of \( \phi \)
\[ \begin{align*}
\phi(r) &= \frac{\pi r^2 \cos \left( \frac{1}{2} \pi r^2 \right)}{2 \sin^2 \left( \frac{1}{2} \pi r^2 \right)} - \frac{1}{\sin \left( \frac{1}{2} \pi r^2 \right)} + \\
\phi^i(r) &= \frac{\pi}{S} \cos \left( \frac{1}{2} \pi r^2 \right)
\end{align*} \] (32)

Figure 2 shows the radial variation of function \( \phi(r) \). Since this represents the effects of viscous damping of the vorticity wave, it shows that the frictional effect grows rapidly larger as the chamber axis of symmetry is approached. This is because the vorticity wavelength becomes shorter (spatial frequency becomes higher) as Figure 1 demonstrates. Therefore the local axial velocity gradient becomes steeper as \( r \) becomes smaller, and the viscous forces become correspondingly more important.

<table>
<thead>
<tr>
<th></th>
<th>L (m)</th>
<th>R (m)</th>
<th>( M_b )</th>
<th>( \delta ) (First mode)</th>
<th>( k_m ) (First mode)</th>
<th>S (First mode)</th>
<th>( \xi ) (First Mode)</th>
</tr>
</thead>
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<tr>
<td>Small Research Motor (Yang and Culick)</td>
<td>0.60</td>
<td>0.025</td>
<td>1.7( \times )3</td>
<td>5.49( \times )4</td>
<td>1.33( \times )1</td>
<td>76.87</td>
<td>1.0309</td>
</tr>
<tr>
<td>Tactical Rocket (Typical Geometry)</td>
<td>2.03</td>
<td>0.102</td>
<td>3.1( \times )3</td>
<td>2.74( \times )4</td>
<td>1.58( \times )1</td>
<td>50.84</td>
<td>0.0624</td>
</tr>
<tr>
<td>Cold Flow Experiment (Shaeffer and Brown)</td>
<td>1.73</td>
<td>0.051</td>
<td>3.3( \times )3</td>
<td>6.07( \times )4</td>
<td>9.24( \times )2</td>
<td>28.30</td>
<td>0.0909</td>
</tr>
<tr>
<td>Space Shuttle SRM</td>
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<td>0.70</td>
<td>2.3( \times )3</td>
<td>1.04( \times )4</td>
<td>6.27( \times )2</td>
<td>27.24</td>
<td>0.0035</td>
</tr>
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</table>
Rotational Velocity Vector Corrections

It is now possible to complete the solution for the unsteady velocity field. The axial component is found by inserting the expressions for $\zeta$ from Eq. (28) and $W(r)$ from Eq. (24) into Eq. (17). The result is

$$\hat{w} = iBr \exp(\phi + i\psi) \sin[k_mz\sin\left(\frac{1}{2}\pi^2\right)]$$

where

$$B^{(r)} = \frac{C^{(r)}}{S} \frac{U_r + \frac{\xi}{S^2} C^{(i)}}{U_r}$$
$$B^{(l)} = \frac{C^{(i)}}{S} \frac{U_r - \frac{\xi}{S^2} C^{(r)}}{U_r}$$

Figure 4 shows the composite axial fluctuating velocity ($w' = \hat{w} + \bar{w}$) and Figure 5 the corresponding phase angle vs. radial position for a typical case.
It remains to determine the radial velocity correction resulting from the production of vorticity. This velocity must be added to whatever surface fluctuation is present due, for example, to pressure or velocity coupled combustion response. It is important to see that this correction is required by continuity since the no-slip condition gives rise to a momentum defect at the surface as in simple boundary layer theory.

In the present case, the continuity equation that must be satisfied by the rotational part of the unsteady flow is

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{v}) + \frac{\partial \bar{v}}{\partial z} = 0 \quad (39)$$

Therefore the radial velocity can be found directly by setting

$$\bar{v} = f(r) \exp(\phi + i\psi) \cos\left[k_m z \sin\left(\frac{1}{2} \pi r^2\right)\right] \quad (40)$$

and using Eq. (36) to represent the axial component. This yields

$$\bar{u} = M_b Br^2 U_i^2 \exp(\phi + i\psi) \cos\left[k_m z \sin\left(\frac{1}{2} \pi r^2\right)\right] \quad (41)$$

where negligible terms (of order $M_b^2$) are not shown.

It is important to observe that this radial velocity fluctuation does not vanish at the burning surface. At $r = 1$, one finds for the real part

$$\bar{u}^{(r)} = -M_b \left( S^2 \left( S^2 + \xi \right) + \xi^2 S^2 \right) \cos(k_m z) =$$

$$= -M_b \cos(k_m z) \quad (42)$$

where the approximate result corresponds to cases for $\xi / S$ small, which it is in most realistic situations. Figure 6 is a plot of Eq. (41) for a typical case. Figure 7 is the corresponding phase angle distribution.

The surface radial velocity correction must be carefully accounted for in the motor stability calculations. To the knowledge of the author it had not been incorporated correctly in any previous stability calculations before those described in Reference 8.

**Effect of Vorticity on System Stability**

It was demonstrated in Reference 8 that the classical flow-turning loss is the one-dimensional equivalent of the volume integral

$$\alpha_{FT} = -\frac{M_b}{2k_m E_m} \iint_{V} (U \times \omega') \cdot \nabla p'_m dV \quad (43)$$

where

$$E_m^2 = \iiint_{V} (p'_m)^2 dV = \frac{\pi L}{2R} \quad (44)$$

is the dimensionless normalization function, evaluated here for a cylindrical chamber with an axial acoustic wave.

As demonstrated in the last section, the processes that lead to the creation of the unsteady vorticity, $\omega'$, appearing in Eq. (43) also give rise to a correction to the radial velocity fluctuation at the chamber boundary. This fluctuation modifies the radial velocity appearing in the familiar driving integral used to represent pressure coupling effects at the surface. Thus there is an additional integral over the burning surface,

$$\alpha_{FT} = -\frac{1}{2E_m^2} \iint_{S_n} n \cdot \bar{u}'_m dS \quad (45)$$

that must be included in the system gain-loss balance if a valid stability assessment is to be made.
Since both stability contributions arise from the vorticity generation process, it is proposed that they be combined to represent the total flow-turning effect. Thus, write for the composite flow-turning stability term:

\[ \alpha_{FT} = \alpha_{FT1} + \alpha_{FT2} \]  

which should be used in both one- and three-dimensional motor stability calculations. The first term, Eq. (43), can be written as a surface integral since

\[ \omega' = \frac{\partial u'}{\partial z} - \frac{\partial v'}{\partial r} = - \frac{\partial w'}{\partial r}, \]  

(47)

The approximate form is valid since the radial velocity fluctuation is small \((r = 0)\) and changes only slowly with respect to \(z\). Also, the integrand in Eq. (43) oscillates rapidly to zero at the centerline \((r = 1)\). Therefore, the value of the volume integral is determined entirely by its upper limit (at \(r = 1\)). The integrand near the upper limit can therefore be accurately represented as

\[(U \times \omega') \cdot \nabla p'_m = - k_m \frac{\partial w'}{\partial r} \sin (k_m z) = \frac{\partial}{\partial r} (\hat{u} \cdot \nabla p'_m),\]  

(48)

since the derivative of the vortical velocity with respect to \(r\) dominates. This is because \(\hat{u}\) involves \(\psi\), whose derivative is proportional to the inverse of the mean flow Mach number as shown in Eq. (22). Then the complete flow turning integral reduces to the surface integral

\[ \alpha_{FT} = - \frac{1}{2E_m} \int \int_{S_R} \left( \mathbf{n} \cdot \mathbf{u}^{(r)} p'_m + \frac{M_b}{k_m} \mathbf{u}^{(l)} \cdot \nabla p'_m \right) dS_b \]  

(49)

where the integration is over the burning surface. Notice that only the rotational part of the velocity fluctuation is involved in the flow-turning integral. Therefore, the flow-turning stability correction vanishes if one assumes an irrotational unsteady flow as was done in formulating the three-dimensional stability algorithm.

Equation (49) is evaluated by inserting the expressions for the velocity components from Eqs. (36) and (42). For example, for a plane acoustic wave, \(p'_m = \cos (k_m z)\),

\[ \begin{align*} n \cdot \mathbf{u}^{(r)} p'_m &= - M_b f \cos^2 (k_m z) \quad (50) \\ \mathbf{u}^{(l)} \cdot \nabla p'_m &= k_m f \sin^2 (k_m z) \end{align*} \]  

(51)

where

\[ f = \frac{S^2 \left( S^2 + \xi^2 + \xi_f \right)}{(S^2 + \xi_f)^2 + (S \xi)^2} \]  

(52)

is determined by the principal scaling parameters. It happens that \(f\) is very nearly unity for any practical case because of the dominant role played by the Strouhal number, \(S\) in Eq. (52). Therefore, the composite flow-turning growth rate contribution is

\[ \alpha_{FT} = \frac{M_b}{2E_m} \int \int_{S_b} \cos (2k_m z) dS_b \]  

(54)

This expression is in dimensionless form, and must be divided by the characteristic time, \(R/\omega_0\), to convert it to the usual dimensional form (in rad/sec).

For a full-length cylindrical grain, the integral of Eq. (54) vanishes showing that there is no net loss of energy from flow-turning for this simple geometry if all effects of vorticity production are accounted for. That is, the loss resulting from acquisition of the axial acoustic motion as gas particles leave the burning zone is compensated by an energy gain due to the additional radial pumping due to the surface momentum defect. Both effects are the direct consequence of the no-slip boundary condition that must be accommodated at the surface.

However, for a partial cylindrical grain of length \(\Delta z\) beginning at \(z = z_0\) (measured from the forward closure as in Figure 8), the net flow-turning contribution is

\[ \alpha^{*}_{FT} = \frac{a_0 M_b}{m \pi R} \left[ \sin (2m \pi (z_0 + \Delta z)/L) - \sin (2m \pi z_0/L) \right] \]  

(55)

in dimensional form. Notice that the net flow-turning effect depends strongly on both the grain geometry and the acoustic mode (mode number \(m\)) being evaluated. Figure 9 is a plot of Eq. (55) for the first axial mode.

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**Figure 8** Geometry for Partial Length Grain.

**Figure 9** Effect of Grain Position on Flow-Turning.
Figures 10 and 11 are similar plots for the second and third longitudinal modes \((m = 2, 3)\). Notice in Figure 10 that the growth rate is zero if the grain length is half the chamber length regardless of its placement. As indicated on the figures, plots end when \(z_0 + \Delta z\) equals the chamber length. For example, the curve labeled “0.6”, ends at \(z_0/L = 0.4\). For the second mode, this plot lies atop the one (shown as a dashed line) for grain length equal to 10% of the chamber length (which ends at \(z_0/L = 0.9\)).

Of the greatest importance in motor stability assessment is that the net flow-turning effect can be either a gain or loss depending on the mode index, \(m\), and the placement and length of the grain.

The situation is even more complicated when there are multiple partial length grain elements. This explains, at least in part, the often confusing experimental results from complex motor grain configurations. End cavities, partial length slots, fins, conocyls and the like have a very definite effect on system stability. For example, for a single grain with half the chamber length, both the first and third modes have no flow-turning loss if the grain is placed at the center. For the second mode there is no flow-turning effect regardless of grain placement as Figures 9-11 indicate.

If a shorter grain is used, say one with a length of only 20% of the chamber length, then the first mode flow-turning effect is stabilizing if the grain is placed in the center of the chamber with \(z_0/L\) in the range \(0.15 < z_0/L < 0.65\). The first mode flow-turning is most destabilizing when this grain is located at the nozzle end of the chamber. The second mode is most destabilized if the grain is placed at \(z_0/L = 0.35\) and is stable if the grain is located at either end of the chamber.

These simple examples illustrate the much greater importance of grain geometry and placement than indicated in earlier stability assessment procedures.

The effect of flow-turning is of decreasing importance as the mode index increases. It can be totally ignored in stability calculations for mode numbers greater than, say, \(m = 10\), in which case the contribution is probably less than the uncertainty in other parts of the stability assessment algorithm.

**Conclusions**

A fully analytical solution to the Navier-Stokes equations describing the unsteady flow in a cylindrical chamber with sidewall injection has been used in this paper to establish the effects of viscosity on the flow-turning effect in rocket motor combustion instability. The solutions are controlled by two scaling parameters. These are the Strouhal number, \(S\), based on the acoustic frequency and the injection velocity at the propellant surface, and a viscous parameter, \(\xi\), based on the inverse of the acoustic Reynolds number.

It was found that the classical flow-turning effect is not significantly affected by viscous forces in any of a wide variety of realistic situations including cold-flow experiments, tactical motors, small research combustors, and very large solid propellant rocket motors.

The great utility of analytical methods in rocket combustion instability assessment is demonstrated by the results of the present study. Many other investigators
have attempted to understand flow-turning by means of numerical techniques. In virtually all of these calculations, the boundary conditions chosen were only partially correct. For the most part, the axial unsteady velocity was correctly set to zero at the surface. On the other hand, if the radial velocity is set either to zero or proportional to an arbitrarily chosen admittance or response function (as it most often is) an important stability element is lost. This approach adequately demonstrates the axial vortical wave motions, but it automatically eliminates the generation of a fluctuating radial velocity correction at the surface. This radial correction must be present to satisfy continuity when the no-slip condition is imposed. Numerical studies must be carefully constructed to account for mass conservation at the burning zone if they are to produce complete information concerning system stability.

It has been proposed in this paper that the influence of the radial pumping due to the surface displacement (or momentum defect) effect be added to the classical flow-turning result. The latter term accounts only for the axial vortical/acoustic energy transfer. The combination is the “composite” flow-turning as defined herein.

This, of course, is only one of several possible interpretations of the results. For example, the radial velocity correction could be interpreted as a “velocity coupling” effect, since it arises only when the acoustic motions are parallel to the burning surface. However, it is not dependent in any way on an interaction of the axial acoustic velocity fluctuations on the combustion processes as in the classical velocity coupling concept. It depends on the acoustic pressure distribution since the latter directly controls the production of unsteady vorticity, which gives rise to the flow-turning effect.

In Reference 8, the relationship to acoustic pressure gave rise to an interpretation of the vortically-generated radial velocity correction as an addition to the pressure coupling term, since the latter depends on pressure in an identical fashion.

Any of the suggested interpretations are certainly acceptable, but confusion can be reduced if the two major effects of vorticity production are combined as suggested here into a composite flow-turning stability integral.

The results demonstrate that the flow-turning can be conveniently represented by a simple surface integral over the chamber boundaries (see Eq. (55)) in keeping with other main elements of the stability assessment algorithm.

The stability results are virtually independent of viscosity, although viscous forces greatly influence the unsteady velocity distributions in the volume of the combustion chamber. This is a consequence of the fact that viscous forces are negligible near the injection surface. They become increasingly important as the chamber axis is approached because the axial velocity gradients become greater due to their dependence on the decreasing spatial wavelength of the vorticity waves.

The composite flow-turning effect can be either stabilizing or destabilizing depending on motor grain configuration. Flow-turning becomes less important as the wave number increases, and can be neglected in the case of higher order \( m > 10 \) (say) longitudinal modes.

The validity of the results can be demonstrated by comparison to actual motor stability measurements. For example, the AEGIS EX72 motor was predicted to be stable by the SSP \( (\alpha = -63 \, \text{sec}^{-1}) \). However, it exhibited first axial mode oscillations at about 300 Hz in qualification testing. Application of the composite flow-turning correction in place of the classical flow-turning predicts an unstable motor with a positive growth rate of \( \alpha = 3 \, \text{sec}^{-1} \).

As a second example, the data set by Harris demonstrates considerable improvement in agreement between prediction and experiment when the composite flow-turning is used instead of the classical one-dimensional flow turning loss. Figure 12 compares the new predictions to the standard one-dimensional SSP calculation for Harris’ experiments. Exact agreement between prediction and experiment is indicated when the data points fall on the diagonal dashed line. Numbers denote corresponding points. The corrected data points cluster around the line of exact agreement.

![Fig. 12 Effects of composite flow-turning on stability predictions (data due to Harris).](image)

Since the new theory indicates a stronger dependence on chamber configuration than earlier stability prediction theories, it can be readily verified by simple laboratory burner experiments in which grain length and position are varied in a systematic fashion with other parameters held fixed.
Finally, much of the controversy surrounding flow-turning and its influence on rocket motor stability has now been resolved. Culick’s original calculation\(^3\)\(^-\)\(^4\) is basically correct and should indeed be incorporated in the three-dimensional stability algorithm. However, it must be supplemented by another term which arises also in the process of vorticity production at the burning propellant surface. This new correction results in a net flow-turning effect that can be either stabilizing or destabilizing depending on mode and chamber geometry.

These new results help to explain the practical observation that more acceptable agreement between experimental data and stability prediction is sometimes achieved when the flow-turning is simply set to zero.

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**References**


