NONLINEAR COMBUSTION INSTABILITY DATA REDUCTION

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Abstract

A crucial element in the practical solution of combustion instability problems in rocket motor development programs is the accurate interpretation of data secured in pulsed motor stability testing. There presently exist no universally accepted or well-established methods for carrying out this type of data analysis. Most investigators apply only linear combustion instability theory; any attempted nonlinear interpretation is then entirely qualitative. Availability of some new high-quality nonlinear experimental data collected in an international collaborative research program presents the opportunity to reexamine the data interpretation methodology currently employed. These data sets contain much information that has not been fully explained, nor have they been subjected to searching scrutiny from the standpoint of nonlinear instability theory. In particular, information of direct value in producing an improved nonlinear instability theory and data reduction procedure is evident in the published material. In this paper, presently accepted nonlinear instability concepts are critically examined from the standpoint of both their correspondence to the observations and their use as diagnostic tools. Suggestions are made for improved experimental techniques and for new directions in the development of practical theoretical models.

Introduction

Despite almost four decades of intensive experimental and theoretical effort, the problem of nonlinear oscillatory behavior in rocket combustors has not been adequately resolved. There are no accepted theoretical models that allow experimental data to be correctly interpreted and the results utilized in formulation of effective corrective procedures. Reasons for the failure to reach the required level of physical understanding are sought in this paper. A major objective is to define new approaches that can eventually lead to appropriate theoretical models. Emphasis is on the use of such models in the interpretive or diagnostic rather than the predictive mode.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\bar{a}$</td>
<td>Mean speed of sound</td>
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<tr>
<td>$k_n$</td>
<td>Wave number for axial mode $n$</td>
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<tr>
<td>$L$</td>
<td>Chamber length</td>
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<tr>
<td>$n$</td>
<td>Mode number</td>
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<tr>
<td>$M_b$</td>
<td>Mach number at burning surface</td>
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<tr>
<td>$p$</td>
<td>Pressure</td>
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<tr>
<td>$\bar{P}$</td>
<td>Mean chamber pressure</td>
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<tr>
<td>$r_n$</td>
<td>Characteristic system amplitudes</td>
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<tr>
<td>$R$</td>
<td>Chamber radius</td>
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<tr>
<td>$z$</td>
<td>Axial position</td>
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<tr>
<td>$\alpha$</td>
<td>Growth rate (sec$^{-1}$)</td>
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<tr>
<td>$\alpha_s$</td>
<td>System stability functions</td>
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<tr>
<td>$\epsilon$</td>
<td>System wave amplitude</td>
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<tr>
<td>$\gamma$</td>
<td>Ratio of specific heats</td>
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Subscripts

- $b$ | Combustion zone |
- $n$ | Mode |

Superscripts

- ($\cdot$)* | Dimensional quantity |
- ($\cdot$)' | Fluctuation |

Availability of high-quality experimental data presents the opportunity to test the validity of key theoretical models on which current understanding of combustion instability is based. This paper utilizes recent data sets collected in an international collaborative research program to determine areas of weakness both in the theoretical models and in their application in the data interpretation process. The focus is on important features of the data that were not addressed in analyses described in papers by Blomshield, et al., Mathes, and Harris. The analyses carried out in those studies were used mainly to test the validity of linear combustion instability prediction algorithms (SSP), since much of the information gathered involved decaying low-amplitude pressure pulses. The data were examined only qualitatively from the nonlinear point of view, since no accepted quantitative theory has been established. In all cases only linear methods of analysis were employed by fitting linear theory to selected portions of the test data. The main objective of these programs, which was to improve understanding of nonlinear instability, was therefore not fully realized.
Background and Approach

The recent test data described above contain much information that was not addressed in the associated reports.\(^1\)\(^-\)\(^3\) In particular, information of direct value in producing a more useful nonlinear instability theory and corresponding data reduction procedures is clearly evident in the published material.

The goal of the work described here is to examine these findings in order to identify a path that could lead to practical theoretical tools. In this process, it is necessary to examine the set of basic assumptions that are used by researchers in nonlinear instability modeling. It will be demonstrated that some of these assumptions may not be consistent with key features of the experimental data. In particular, questions related to behavior of pulse-triggered oscillations with shock-like waveforms must be addressed. Also, the presence of multiple limit cycle amplitudes are clearly indicated in the data, and this behavior must be incorporated into the nonlinear modeling.

A feature that greatly simplifies the extraction of nonlinear mechanism information is that the experimental waveforms display very nearly fixed spectra over significant periods of time, over large variations in oscillation amplitude, and for large numbers of cycles of oscillation. Therefore, theoretical models based on modal energy transfer might not be required.\(^4\)\(^-\)\(^6\) The focus of current theories on mode coupling must therefore be carefully considered. Under the conditions described, limit cycle oscillations, mean pressure shifts, and triggering behavior appear to be controlled by nonlinear energy gain and loss mechanisms associated with flow and combustion processes in the presence of a shock-like steep fronted systems of waves.\(^7\)\(^-\)\(^9\)

Methods used by previous investigators in extracting the linear stability information from the nonlinear decay data are carefully examined. For example, in the NAWC data,\(^1\)\(^-\)\(^2\), there is evidence that pulses decayed in nearly every case to a stable limit cycle rather than to a state of zero oscillation. Only in the latter situation does the growth rate data give a valid estimate of the linear decay alpha.

The capability to fit a straight line to a selected portion of the data (plotted in logarithmic form) does not ensure that one is operating in the linear range. In fact none of the data appear to follow linear behavior for periods of time comparable to the decay or growth of a pulse to an equilibrium state. Triggered behavior observed after larger pulses with subsequent approach to a high-amplitude limit cycle substantiates the difficulties accompanying this method of data reduction.

Linear decay growth rates found in the manner just described (first longitudinal mode only) from selected examples of the pulse decay data were plotted against predictions from the Air Force Standard Stability algorithm (SSP\(^12\)\(^-\)\(^13\)) in the NAWC program. A reasonable correlation seems to result. This led the investigators\(^1\)\(^-\)\(^3\) to conclude that the SSP is a valid representation for the system behavior. This key conclusion is challenged in this paper.

It is demonstrated in what follows that the linear growth rates are positive instead of negative in much of the data taken by Blomshield and his team. Negative linear growth rates are inconsistent with key elements of the data set. This indicates that there are stability elements missing in the linear stability algorithm as it was applied in the study.

The results of Harris were improved by incorporation of corrections to the linear theory deduced in recent studies.\(^14\)\(^-\)\(^16\) These studies have shown that there exist linear stability effects related to physical mechanisms not represented in the current code. These are for the most part related to rotational flow field effects; the SSP is based, for the most part, on irrotational fluid mechanics, since it assumes that the gas motions are acoustic waves.

Other features of the test configurations indicate that flow-driving effects related to formation and propagation of large-scale vortex structures must also be considered in the linear stability modeling.\(^17\)\(^-\)\(^19\)

In this paper, the data sets described above are carefully evaluated. After a brief review of currently available theories, the data are used to check the assumptions on which these models are based.

Consideration of elements of modern nonlinear oscillator theory are also introduced to clarify key difficulties with the current theoretical models for nonlinear combustion instability. Effects of “self-organization” of nonlinear systems leading to solitons or shock wave solutions are reviewed.\(^20\)\(^-\)\(^23\)

Of great significance is the evidence of shock waves in the experimental data.\(^24\)\(^-\)\(^26\) The compatibility of shock wave motions with the acoustically based models now in use must be established. In fact, it is readily demonstrated that the majority of current models are basically theoretical descriptions of the steepening process by which initially small amplitude wave motions transition into shock waves. Much of the nonlinear behavior predicted in such models results from the transfer of energy from lower modes to the higher harmonics. This implies a net energy loss since the process describes, in effect, the creation of entropy.

It is ironic that some of the earliest nonlinear models were based on recognition of the central role played by oscillating shock waves.\(^22\)\(^-\)\(^23\) It will be demonstrated here that a return to this earlier view can lead to both improved physical understanding and to more simple and practical methods for reduction of experimental data.
Review of NAWC Experimental Results

In what follows, attention is focused on the excellent data set described by Blomshield, et al. This program involved careful testing of twenty motors with a configuration typical of operational tactical rocket systems. Each test could be pulsed up to three times by pulses of preselected amplitude. A typical pressure amplitude vs. time trace is shown in Figure 1.

This particular plot (Motor Test No. 4, star forward geometry, 1000 psi mean pressure) illustrates several familiar nonlinear features that characterize the test series. The first pulse (decay) occurs at 0.97 seconds and has an amplitude of 43 psi. This pulse is clearly visible in Fig. 1. The second smaller pulse (25 psi) occurs at 1.96 seconds and leads to triggered growth to a high-amplitude limit cycle \( (\varepsilon = p'/\bar{P} = 0.59) \) accompanied by a large DC shift which persists for about 1 second. The nonlinear oscillations then decay as the motor taildown begins. The behavior depicted in Fig. 1 was repeated in tests involving motors with full-length aft star, full-star, and cylindrical geometries. Both the initial (low-amplitude) and the triggered (high-amplitude) limit cycle amplitude was quite reproducible for tests with similar propellant and geometry.

This indicates that although nonlinear oscillations are clearly present, the test outcomes did not display chaotic behavior. In fact, there is much evidence of self-organization, which is a mode of behavior that is often encountered in nonlinear oscillators. Further discussion of this interpretation will be presented in later sections.

Nonlinear aspects of the data are similar to those experienced in other nonlinear instability studies. For example, the mean pressure shift data is strongly correlated to the limit amplitude of the accompanying pressure fluctuations. This correlation is described in terms of a linear least-squares fit in the NAWC report. However, as shown in Fig. 2, a quadratic fit is superior. This is also in better agreement with nonlinear instability theory, which indicates that the DC shifts are a feature of nonlinear interactions proportional to the oscillation amplitude to the second power or higher.

The dominant frequency of both the low- and high-amplitude limit cycle oscillations corresponded closely to the first longitudinal acoustic mode. This observation is often used as the basis for theoretical models based on superposition of acoustic waves. Despite their low amplitude, the waveforms of the decaying oscillations produced in first pulse (pulse just before \( t = 1 \) in Fig. 1) show the same nonlinear attributes (steep wave front, etc.) present in the later triggered oscillation.

![Fig 2. - Correlation of DC Shift with Amplitude](image)

Spectral Data for Untriggered Motor Operation - Low Amplitude Limit Cycle

An important feature of the NAWC data set is the presence of a low-amplitude limit cycle oscillation in cases where triggered high-amplitude oscillations were not present. Figure 3 shows waterfall data for a motor of this type (Motor No. 5, Star-Aft, 1000 psi planned chamber pressure). In this motor only the first pulse was fired. The pulse occurred at \( t = 0.965 \) with an amplitude of 73 psi. This pulse amplitude was of sufficient strength to cause triggered instability had it been fired at a later point in the burn. Typically, pulses fired after mid-burn were likely to produce triggered response if of sufficiently high amplitude. There is little evidence of the decaying pulse in Fig. 3 because it has very nearly dissipated at the closest time-slice \( (t = 1.0 \text{ sec.}) \). The small second
and third harmonic activity appearing at this time is probably related to the pulse.

The presence of a low-amplitude (less than 2 psi) oscillation at the first longitudinal acoustic frequency (approximately 320 Hz) is evident especially toward the end of the burn. Low-amplitude second and third harmonics accompany this feature.

**Fig 3. - Waterfall Plot for Motor No. 5**

There is also some evidence of electronic noise. 60-cycle oscillations and some of the harmonics (especially the 180- and 240-Hz harmonics) are clearly discernible. This seems to answer questions raised by other investigators as to the origin of the low frequency data. It was suggested that they arise from "electronic noise." However, the waterfall plots for Motor No. 5 and similar plots for other motors in the test program make it clear that a real chamber gas oscillation was present even in the absence of pulsing. It is obviously imperative to understand the significance of these oscillations in the context of the nonlinear behavior of the system.

The position taken by the NAWC investigators was that "...the oscillations observed are due to processes internal to the motor...although the spontaneous oscillations which occur naturally in this motor might be of concern in some real-world motor applications, they were not considered a subject worthy of intense study in this program." The present writer could not disagree more with this conclusion. Any spontaneous oscillations from whatever source are of the greatest importance because they indicate that the motor system is linearly unstable. In applying the SSP in the data reduction, it is necessary that all system gains and losses are accounted for, even if they are related, as the report suggests, "...to naturally occurring acoustic oscillation due to turbulent fluid flow."

Clearly, no realistic nonlinear data interpretation can be undertaken without first achieving the most complete possible understanding of the motor operation from the linear standpoint. This is the reason that so much effort has been devoted to development of linear theories of combustion instability despite the fact that the most important practical problems involve nonlinear behavior. In addition to the classical linear effects such as pressure coupling, nozzle and particle damping, and so on, it is often necessary to account for vortex shedding, turbulence, and distributed combustion effects in the linear modeling. The likelihood of flow-driven oscillations in the NAWC motors is discussed in more detail in the next section.

**Low-Amplitude Pulse Tests**

Many of the pulses in the NAWC program were fired with amplitudes below the critical triggering amplitude at the corresponding time during motor burn. A typical decaying pulse is shown in Fig. 4. Notice that it is visually evident that the pulse does not decay to zero amplitude. The presence of a low-amplitude oscillation both before the pulse and after its subsidence is clearly evident. This feature raises many questions regarding methods used to reduce the data.

**Fig 4. - Typical Decaying Pulse (Motor No. 3)**

In the NAWC program, the assumption is made that decaying pulses can be treated by means of linear instability theory. This implies that the linear growth rate for a particular mode can be deduced in the familiar manner by hand fitting a straight line to a logarithmic plot of the data as depicted in Fig. 5. It happens that decay data determined in this fashion agree fairly well with predictions made using the SSP code. Figure 6 shows these results. It is noteworthy that agreement was especially poor for the half-length motors tested in the program. Also, no attempt was made in the SSP calculations to account for vortex shedding effects or for recent corrections to the important flow-turning loss. Thus, the conclusion that the test results verify the validity of the uncorrected SSP calculations must be questioned.
High-Amplitude Pulse Tests

A typical high-amplitude pulse and the resulting triggered response is shown in Fig. 1 and in more detail for another motor in Fig. 7. In the latter figure (Motor No. 2, high-frequency transducer PK-102) the presence of low-amplitude oscillations before the triggering pulse are readily discernible. The chaotic motion during the pulse is rapidly damped and an organized transition to the limit cycle commences. The presence of steep wavefronts is evident in Fig. 7; all frequencies up to 10 kHz are represented.

Since current theories are based on the concept that the transition from the initial state to the final limit cycle oscillation is controlled by energy shifting between the modes, it is important to examine the data in the frequency domain. Figure 8 shows a waterfall plot for pulse No. 1 fired at t = 0.745 sec. during Motor Test No. 9. The energy peaks centered around the 17th axial mode are probably the effects of resonant oscillation of the pulser cavities, which remain open after pulse firing. These high frequency oscillations are present in all the high-amplitude motor spectra.

It is vital to understand that the time range in Fig. 8 encompasses the entire transition process, not just the final limit cycle oscillations.

Spectral content of the wave motions was remarkably constant throughout the system state transition. To emphasize this point, Fig. 9 shows a superposition of the spectral data for each time slice with the amplitude normalized to the corresponding first mode amplitude. It is of the greatest significance that there is no evidence of major changes in the relative modal component amplitudes, at least for the first few modes. Remember that it is these modes that contain the vast majority of energy in the system oscillations. It is expected that there would be more and more variation with time for higher-order modes, especially if they represent resonant oscillation of the pulser cavities as already suggested. Therefore changes in relative amplitude are not controlling the triggered system transition to the high-amplitude limit cycle for the main part of the combustion chamber. The impact of this observation on theoretical modeling must be addressed. Initial considerations are described in a later section.
Interpretation of Data Using Nonlinear Theory

The objectives of this section are to briefly trace the origins of nonlinear combustion instability theories and to examine their relationship to the data presented above.

In particular we seek to understand several unusual features of the data that have not been discussed previously. Any useful theory must encompass both these new findings as well as earlier observations. The following items must be accounted for: (1) multiple limit cycles, (2) nearly invariant modal content during triggered growth, and (3) spontaneous transition from a triggered limit cycle to a yet higher amplitude limit cycle oscillation. The latter effect is not depicted in any of the data shown in the summary of the last section, but appears in the data for Motor firing No. 8b in the NAWC study.

In fact all of these features have appeared in earlier experiments. For example they were first observed by B. Hussey in tests of a tactical rocket operating under conditions similar to those in the NAWC tests. Limit cycle oscillations with amplitude of approximately $\varepsilon = p' / P = 0.03$ began during the ignition transient and persisted until the system underwent a spontaneous transition to a much higher limit amplitude of $\varepsilon = 0.15$. In another test of this device a triggering pulse initiated a transition to a limit amplitude of $\varepsilon = 0.5$, which is similar to amplitudes observed in triggered limit cycles in the NAWC data. Also, the spectral content of the waves did not vary significantly over the entire period of nonlinear oscillation.

Any valid theory of nonlinear combustion instability must be capable of yielding a physical understanding of effects such as those described. It is imperative to attempt to understand the observed behavior since it appears to represent characteristics typical of operational rocket systems.

We must now examine the origins and the working features of nonlinear instability models to determine if they properly represent the observations.

Origins of Nonlinear Instability Theory

Origins of current theoretical concepts can be traced to the earliest studies of combustion instability. It was recognized that nonlinear effects must play a central role in the phenomenon, especially in the important problem of longitudinal instability. It is ironic that the first detailed analyses of this sort were based on the idea that shock waves must be involved. This was verified in experiments by Brownlee in which the presence of shock waves were established by means of Schlieren flow visualization. Analytical approaches followed Lighthill's methods using characteristic coordinates. An interesting deductive approach introduced by Chester was also applied, to good effect by Mitchell and others.
Unfortunately, this promising technique was not developed in the detail necessary for its application in solid rocket motor instability with internal burning grains. In general, the methods just described are based on the idea that shock waves appear naturally when resonant oscillations are present in the natural chamber acoustic frequencies.

Another approach appeared during the late 1960s. At this time there was intense development in large solid rocket systems, and the work was motivated by the need to solve longitudinal mode instability problems in highly aluminized propellants. Several research studies were carried out mainly with Air Force funding. The work by Beckstead and Jensen\textsuperscript{10} was a typical example. During this program, Beckstead found that a simple extension of the ideas of linear combustion instability theory seemed to result in more acceptable correlation with experimental data. His model was based on the addition of a nonlinear term to the usual linear model

$$\frac{de}{dt} = \varepsilon \alpha$$

Beckstead's "\( \pi \) correlation" extended the model to second order by the addition of term proportional to the square of the oscillation amplitude:

$$\frac{de}{dt} = \alpha e + \beta e^2$$

This model formed the basis of extensive attempts to analyze experimental data by using formulas of this type in a curve fitting mode to arrive at the governing parameters such as \( \alpha \) and \( \beta \). Unfortunately, it was found to be impossible to find consistent results in this fashion, and the method was abandoned.

This failure motivated Culick to attempt a new nonlinear modeling effort based on an extension of the acoustic wave models used in linear combustion instability theory. The main working hypothesis was that since spectral analysis indicates that frequencies closely matching predicted acoustic natural modes dominate the data, then such modes represent an appropriate solution base. This synthesis of the classical mode shapes leads to a basic representation of the unsteady pressure field as

$$\frac{p'}{p} = \sum_{n=1}^{\infty} \eta_n(t) \psi_n(r)$$

where \( \psi_n(r) \) are the classical mode shapes and \( \eta_n(t) \) are time varying modal amplitudes. This is the Galerkin approach, and when Eq. (3) is substituted into the nonlinear wave equation and boundary conditions governing the chamber motion, a set of coupled oscillator equations results. Solutions of these equations then yield information on the system nonlinear behavior. Details are discussed in the excellent survey paper by Culick.\textsuperscript{5}

**Theory Based on Interacting Acoustic Modes**

Models of the sort described in Eq. (3) have been extensively studied for several decades. They are attractive because they appear to offer a simple extension of the acoustic models that are the foundation of the linear theory. An important feature of this representation is that it can be interpreted as a set of nonlinear coupled oscillators. This "mode coupling" effect is then the focus in attempts to explain observed nonlinear behavior.

What is not clear in the implementation of this method is the ultimate state of the initial ensemble of oscillators. If cascading of energy from mode to higher mode is the basic nonlinear mechanism, then it appears that the final state of the system should be chaotic. However, studies of similar sets of nonlinear oscillators show that another more interesting outcome is likely, namely the self-organization of the initial energy into a synchronized motion. In dispersion free systems such as those arising in the present problem: the outcome is shock wave motions.

Since many modes are required in adequately representing the waveforms seen in data (for example, examine the spectral data in Figs. 8 and 9), then in applying this method in practical situations, one is faced with the need to fit a large set of time-varying amplitude functions to a set of time data found presumably by spectral analysis of the data.

It is interesting that Culick's original motivation for developing this approach was based on the failure of curve fitting as attempted by Jensen.\textsuperscript{22} Yet, it appears that in using the mode coupling method in the diagnostic mode wherein quantitative results are required, just such curve fitting is then unavoidable.

This is probably the main reason that few attempts have been made to apply the theory directly to actual experimental data. In fact, even numerical experiments with the method are usually limited to a small number of modes because of the great complexity arising in a more realistic ensemble.

In the next subsection, we examine an alternative interpretation of the observations. In studying the formulation of the mode coupling based analysis, it is clear that the basic process being modeled is the transition of the system into an organized system of shock waves. This suggests a return to ideas that motivated the older nonlinear analyses.

**Theories Based on Shock Wave Motion**

Application of ensembles of nonlinear oscillators are a common stratagem in solving nonlinear problems. It is therefore important to become familiar with the solutions generated in other fields, since this information yields useful insight into the present application.

Suppose the system is represented (as in Culick's model) by a chain of coupled nonlinear oscillators. These represent the acoustic modes of the chamber. In view of
the absence of dispersion for acoustic waves, the spectrum of the eigenfrequencies of the resonator will be equidistant. Frequencies are integer multiples of each other. For a typical weak nonlinearity of the chamber gas flow, which is quadratic in the amplitude to first approximation, energy initially concentrated in a single mode will cascade to higher and higher order modes.

The energy is gradually pumped into higher and higher frequency oscillations. An intuitive guess about the future of the system would be extremely difficult. This holds even for the excitation of a larger and larger number of oscillators or modes with very weak nonlinearity. The effect of a small nonlinearity of a large number of oscillators gives rise to indefiniteness. Most data of the sort described in the last section shows no tendency to chaos. Instead, there is much evidence of organized behavior.

To resolve this indefiniteness without an a priori hypothesis it is inevitable to solve the initial nonlinear problem outside the realm of quasi-linear physics. Some possible ultimate behaviors of the system are:

1. After a sufficiently long time, nonlinear frequency shifts will lead to loss of the resonant relationship between the modes, and the system will return to an ensemble of uncoupled weakly nonlinear oscillators with independent frequencies.

2. The phases of all oscillators will be synchronized and a shock-like organization will develop.

Considerations such as these have led to much success in solution of nonlinear problems in other fields. Again, the key to success is the recognition that nondispersive wave systems like those represented by the present problem are quite often characterized by "self-organization," that is by the generation of shock waves.

To summarize, the gas motion observed in pulse triggered instability is not stochastic. It is characterized by a well-defined transition to a steady limit cycle. The spectrum is synchronized or "frozen" throughout the transition. This is the result of the rapid formation of a shock wave during the pulsing process.

A Simplified Shock Wave Model

These ideas led to the nonlinear energy balance analysis by Flandro. This method was an attempt to bridge the gap between the Galerkin approach used by Culick, and the earlier shock wave models. It is strongly based on the experimental observation that in pulsed systems, the creation of a shock structure is the most likely outcome. Then fast simplifications result. The starting point is Eq. (3) modified in such a way that the modal components are synchronized in the fashion suggested by the experimental data (see Figs. 8 and 9). Then only one amplitude function is required. This could, for example, be the amplitude of the first mode. Then put

\[
\frac{p'}{p} = \varepsilon \sum C_i \eta_i \sin(\omega_i t + \phi_i) \tag{4}
\]

and the relative amplitudes \(C_i\) and the corresponding phase angles \(\phi_i\) characterize the particular waveform. It has been demonstrated that this model, consisting of a set of standing classical acoustic waves can be used to accurately represent a traveling shock wave, or even a set of shock waves. The reader is referred to Reference 8 for the details.

The function, \(\varepsilon\), which changes slowly with time, is now interpreted to be the global system amplitude representing the evolution of the system as it grows or decays in time. Mode coupling analyses seek to track the evolution of each of the mode shapes \(\eta_i(r)\) as functions of time. This steepening process occurs so rapidly in the presence of a pulse that it is effectively accomplished during the brief period in which the pulse is accommodated by the wave system. These waves are traveling shock fronts. It is noteworthy that they can be represented analytically as a superposition of standing acoustic modes.

Use of Eq. (4) is only justified under the conditions stated, namely when mode synchronization is appropriate. Then if \(\varepsilon\) is treated as a slowly varying function of time, the nonlinear wave equation yields a first-order nonlinear differential equation for the global system amplitude. In general form, one finds

\[
\frac{1}{\varepsilon} \frac{d\varepsilon}{dt} = f(\varepsilon) \tag{5}
\]

where \(f(\varepsilon)\) is a complex nonlinear function of amplitude and a multitude of flowfield and geometry parameters.

It changes slowly with time. Notice that this nonlinear equation has the same form as the linear amplitude equation (Eq. (1)). In keeping with the perturbation expansion process which forms the basis for the entire combustion instability theory, it is natural to extend the linear theory by adding higher order terms in powers of the global amplitude. For example, for a fourth-order system, put

\[
\frac{1}{\varepsilon} \frac{d\varepsilon}{dt} = \alpha_1 + \alpha_2 \varepsilon + \alpha_3 \varepsilon^2 + \alpha_4 \varepsilon^3 = \frac{-\alpha}{\alpha_1(r_1 - \varepsilon)(r_2 - \varepsilon)(r_3 - \varepsilon)} \tag{6}
\]

where the \(\alpha_i\) are functions of chamber parameters as in the linear case; these are slow functions of time. \(\alpha_1\) is therefore equivalent to the linear growth rate. The \(r_i\) are critical points of the system, which we will associate with limit cycle and triggering amplitudes.

System Phase Plane Diagrams

A major benefit of the formulation shown in Eq. (6) is the ease of interpretation in terms of phase plane
Such diagrams are quite difficult to construct in equivalent mode coupling models discussed earlier. Second- and third-order versions of this equation have been studied extensively. The second-order version allows either a triggering point or a limit cycle but not both. The third-order form gives one limit cycle and one triggering point. Figures 10 and 11 show the general features of these two formulations. The second-order results are the equivalent of Beckstead’s PI correlation. He utilized a form of the third-order result in his qualitative description of the features of the NAWC data.

Fig 10. - Second-Order System Model (equivalent of Beckstead PI Theorem)

Fig 11. - Third-Order System Model
Extending this idea leads one to expect that at least a fourth-order system (with three critical points) is required to accommodate data of the sort observed in the NAWC testing. That is, one must account for two limit cycles and an intermediate triggering amplitude.

Flandro\textsuperscript{8,9} utilized a fourth-order model in which explicit expressions for the $\alpha_i$ were derived to account for a pulse decay in an inert chamber. This model produced excellent agreement with the experimental pulse decay data by Lovine.\textsuperscript{26} The former study also included a first attempt to derive formulas (to the third-order) for the $\alpha_i$ for actual motor systems.

Figure 12 describes the characteristics of the fourth-order system description. Two cases are shown. The one on the left illustrates the behavior for a system with a negative linear growth rate. Notice that this system is characterized by two triggering points with an intermediate limit cycle point. The case shown on the right is for a positive linear growth rate. This shows the possibility of two limit cycles with an intermediate triggering amplitude. This closely resembles the behavior seen in the NAWC data set, and hence represents the lowest order system description that describes the observed behavior.

Notice that other outcomes are possible depending on the values of the four system parameters, $\alpha_i$. For example, one can find cases for which there are fewer (or even no) real roots of the right hand side of Eq. (5). These are not of interest in the present situation. They undoubtedly describe other situations that have been experienced in real motor testing.

**Application of Second-Order Theory to Pulse Decay**

Since the amplitudes are generally quite small in cases involving decay of a pulse, it appears possible to neglect the higher order corrections. Then a method similar to that introduced by Beckstead\textsuperscript{10} results. However, we do not propose here to employ general curve fitting methods as used in the latter work. Notice that in Eq. (2) there are two unknown functions to be determined from the data. If curve fitting is used in an attempt to determine these functions, it is readily demonstrated that no unique results are forthcoming. The problem here is that not enough of the experimental information is accounted for. Knowledge of the limit cycle amplitude is the key. Figure 10 illustrates the approach. Notice that the right-hand figure describes the situation encountered in the NWAC tests. Pulses were generally larger than the observed limit cycle amplitude, so the
experiments correspond to the pulse labeled Pulse 2 in Figure 10. The system then decays to the limit cycle as shown rather than to zero amplitude. The linear growth rate must be positive. This additional information then allows an unambiguous least-squares fit to the data Figure 13 shows typical results.

![Graph](image)

**Fig 13. Application of Second-Order Model**

Two cases are shown illustrating the effect of the limit amplitude, \( r_f \), used in the calculations. Values for the parameters are, as expected, quite sensitive to the limit amplitude data. Therefore, it is imperative in using this method to devote special attention to the limit cycle measurements. Notice that the use of the second-order method allows a superior representation of the entire data set describing the pulse. The linear method requires selection of only some of the points which appear to lie in a straight line.

The results show that the linear growth rate is positive. This shows that the SSP calculations used in the NAWC report did not incorporate all of the operative gain and loss mechanisms. In particular, effects of vortex shedding are not represented.

**Effects of Vortex Shedding**

A brief discussion of additional energy sources leading to the observed linear instability in the NAWC tests is appropriate. The report indicated the belief that flow effects such as turbulence might explain the low level "background" oscillations. Earlier studies have indicated that flow-driving from the formation of large scale vortex structures is a far more efficient energy source than coupling to broadband turbulent flow sources.

Examination of the motor configurations used in the tests suggests that the geometries would promote vortex shedding. Since the grain ends were not inhibited, the aft edge of the propellant charge gradually moves forward during the burn. The rapid increase in chamber cross-sectional area leads to flow separation with the attendant vortex formation. Intensive theoretical, numerical and experimental study of such interactions are underway by several groups seeking to find improved predictive models. Currently available models are useful in gaining a qualitative understanding of the phenomenon.\(^{17,18}\)

**Application of Fourth-Order Theory to Triggered Pulse**

In applying the shock wave model to the reduction of the NAWC pulse triggered data set, it is necessary to construct an algorithm that determines the four required values of the \( \alpha_n \). This is facilitated in this case by the availability of several important elements that can be determined from the test data: 1) Amplitude of the low-order limit cycle, 2) amplitude of the triggered limit cycle, and 3) sets of amplitude vs time data, \( e(t) \) from the pulse measurements. In the following discussion, only a preliminary method is described. Much improvement will be possible if experiments are deliberately designed to provide the needed data. Again, reliance on curve fitting is to be minimized.

Figure 12 is a phase plane illustration of the system using a fourth-order representation. Again the right-hand figure describes the situation since the linear growth rate is positive. The two limit cycle amplitudes, \( r_f \) and \( r_3 \), can be deduced from the measured data.

In particular, in the simple example discussed here, the changes in system parameters between two pulses, one decaying and one triggered, will be neglected. It must be understood that this is simply and expedient to illustrate the technique. To achieve accurate results, several identical motor tests should be run, with the only variable being the pulse amplitude at a single selected point in the motor burn. Then the data from a growing and a decaying pulse would correspond to the same motor configuration and the outcome would properly represent the motor behavior at that time.

Then, following the approach used in linear analysis, we integrate the right side of Eq. (6) with respect to time, which yields the four integrals:

\[
\begin{align*}
I_1 &= \int_{t_1}^{t_f} dt = (t_f - t_1) & I_2 &= \int_{t_1}^{t_f} e(t) dt \\
I_3 &= \int_{t_1}^{t_f} e^2(t) dt & I_4 &= \int_{t_1}^{t_f} e^3(t) dt
\end{align*}
\]

(7)

taken over the duration of the pulse, \( t_f - t_1 \). Care must be used to start the integration at an initial time which excludes data distorted by the "smearing out" or adaptation process of the pulse. The initial time is usually readily identified by examination of the amplitude vs time.
data plots. The final time is chosen so that a sufficient number of data points are taken to characterize the pulse as it either decays to a low-order limit cycle or grows to a triggered limit cycle.

Notice that no distinct trigger point can be discerned in the present data sets. Such information can only be secured experimentally by systematically incrementing the pulse amplitude in a set of several tests of identical motors. This would most likely be a costly approach, so we will rely here on the data analysis to supply this information.

Finally, one can write a set of five simultaneous equations for the five unknowns:

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \text{ and } r_2$$

This version of the method is based on the availability of at least two fairly closely timed pulses so that the integrals of Eq. (6) are carried out for two sets of conditions. This yields two equations of the form:

$$\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4 = \ln \left( \frac{e_i}{e_f} \right)$$

(8)

The remaining three equations

$$\begin{align*}
\alpha_1 &= -\alpha_2 (r_1 r_2) \\
\alpha_2 &= \alpha_4 (r_1 + r_2 + r_3 + r_4) \\
\alpha_3 &= -\alpha_4 (r_1 + r_2 + r_3)
\end{align*}$$

(9)

represent relationships between the turning points and the four \(\alpha_i\) functions to be determined. The set of five equations can then be solved algebraically for the required system information.

The method described does not employ numerical differentiation nor curve fitting of the data. The preliminary results show great promise in nonlinear data reduction. Application to a typical pulse is illustrated in Fig. 14.

**Fig. 14** - Typical Triggered Pulse Interpretation (Motor No. 3, Pulse 2)

**Conclusions**

The NAWC nonlinear experiments are characterized by multiple limit cycle amplitudes and linearly unstable oscillations. Hence, the conclusion stated in the NAWC report that the results establish the validity of the linear stability approach must be questioned. However, it is significant that the growth rates measured are in the correct order of magnitude and they appear to change during burn in the manner predicted by the theory. This leads to the conclusion that the linear theory is surely operating correctly, and that the reason for SSP calculations not exactly fitting the experimental data is to be found in elements such as vortex shedding and other corrections that have not been included in the SSP code in its present form.

The authors of the NAWC report attribute the low-level oscillations to "noise" related to turbulence. However, in addition to the effects of unsteady rotational gas flow described in Refs. (14-16), it seems more likely that organized vortex shedding could be an additional source of driving energy not included in the Standard Stability Analysis. These energy sources are far more efficient in driving organized system pressure oscillations than broadband turbulence. In fact, it now appears that the vortical effects described in recent studies represent the first stage in transition to turbulence in the motor chamber.16, 17

The low-amplitude nonlinear oscillations present in all of the tests must be interpreted along with the other system features including the pulse decay or growth data. It is not correct to treat the low-level oscillations as an isolated feature such as back ground noise. Their presence has vitally important consequences in the proper interpretation of the complete data set.

These conclusions are not to be interpreted as a negative assessment of the NAWC work. The writer has examined much data of this sort in the past, and there is no question that this data set is by far the best of this type available in the open literature. There is much evidence of painstaking care in the experimentation and in the handling of the data. Application of the linear method of data reduction can of course be justified on the basis that it was the only method available to the investigators. This emphasizes the great need for continued work in the nonlinear theory, especially in terms of development of practical analytical tools.

Again, it must be emphasized that no criticism of existing theoretical approaches is intended. These works have been of great value in the exploration of this difficult problem area. What is sought in the present effort is the means to utilize these models in the most efficient and practical way. This can only be done by exploiting features such as the fixed waveform behavior resulting from formation of shocks, especially in pulse triggered cases.
In the latter regard, it must be understood that the presence of synchronized waveforms does not imply that there is no energy flow between the modes. It means only that the net transfer to a given mode is effectively zero. That is, the quantity of energy transferred into a particular mode from all sources is exactly cancelled by the net transfer out of it during this special type of gas motion. Also, it is important to observe that there is always a system energy loss when shock waves are present. This loss is not well represented in analyses based on superposition of acoustic modes. That is, in the absence of shocks, the nonlinear interactions do not change the net energy contained in the system at a given time (assuming there are no external energy sources or sinks); they only cause a rearrangement of the frequency distribution of the energy. However, the associated loss is readily computed by classical methods of shock wave analysis.8

The evidence of shock waves in the tests with triggered growth suggests a simplified method of data analysis that was tested in a preliminary way in the course of the work described here. Nonlinear "self-organization" of the system leads to a traveling shock system that can be represented by a superposition of acoustic modes with fixed relative amplitudes. Then it is justified to use a simple fourth-order extension of the linear stability theory, which provides a practical method for carrying out data reduction for nonlinear systems with multiple limit amplitudes.

Further work on the modeling of the stability functions, $\alpha$, bringing into the problem actual representations of the various nonlinear gasdynamics and combustion processes seems to be justified. Theoretical representations that admit shock behavior at the outset23, 25 will finally yield a method that best fits observations of the type described in this paper. Eventually, this approach will give rise to a useful predictive algorithm.

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References


11Beckstead, M. W. personal communication.


14Brownlee, W. G., "Nonlinear Axial Mode Combustion

15Levine, J. N., Baum, J. D., and Lovine, R. L., "Pulse
Triggered Instability in Solid Propellant Rocket Motors:
A Numerical and Experimental Study," AIAA J., Vol. 22,

16Flandro, G. A., "Effects of Vorticity on Rocket
Combustion Stability," Journal of Propulsion and Power,

AIAA Joint Propulsion Conference, San Diego, CA, July
1995.

18Flandro, G. A., "Effects of Vorticity Transport on Axial
Acoustic Waves in a Solid Propellant Rocket Chamber"
Proceedings of the 1989 ASME Annual Meeting, San

19Flandro, G. A., "Vortex Driving Mechanisms in
Oscillatory Rocket Flows," Journal of Propulsion and

20Flandro, G. A., "Assessment of Flow Driven Pressure
Oscillations in Shuttle ASRM Rocket Motor," Final
Technical Report, Contract W31014-84-D-0040, Aerojet ASRM
Division, Idaho, MS, October, 1993.

21Brown, R. S., et al., "Vortex Shedding as a Source of
Acoustic Energy in Segmented Rockets," Journal of
Spacecraft and Rockets, Vol. 18, No. 4, 1981, pp. 312-
319.

22Fox, P. A., "Perturbation Theory of Wave Propagation
Based on the Method of Characteristics," Journal of
Spacecraft and Rockets, Vol. 18, No. 4, 1981, pp. 312-
319.

23Chester, W., "Resonant Oscillations in Closed Tubes," J.

2, No. 7, July 1964.

25Mitchell, C. E., Crocco, L., and Sirignano, W. A.,
"Nonlinear Longitudinal Instability in Rocket Motors
with Concentrated Combustion," Combustion Science and

26Flandro, G. A., "Nonlinear Combustion Instability

27Vuillot, F., "Vortex-Shedding Phenomena in Solid Rocket
Motors," Journal of Propulsion and Power, Vol. 11, No. 4,