INSTABILITY ON NONLINEAR COMBUSTION

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ON NONLINEAR COMBUSTION INSTABILITY

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Abstract

A crucial need in rocket motor development programs is a valid analytical/computational model of nonlinear combustion instability. The motor designer frequently must attempt to estimate not only the probability that a particular acoustic mode may "tend to grow," but what will be the amplitude of the oscillations if they are triggered in some way. A major objective of nonlinear modeling must be to estimate the system limit amplitudes as well as triggering amplitudes above which a pulse containing oscillatory energy causes the system amplitude to increase to a higher-amplitude limit cycle.

Along with this is a need for accurate interpretation of data secured in pulsed motor stability testing. There presently exist no universally accepted or well-established methods for carrying out this type of data analysis. Most investigators apply only linear combustion instability theory; any attempted nonlinear interpretation is then entirely qualitative.

In this paper, presently accepted nonlinear instability concepts are critically examined from the standpoint of both their correspondence to the observations and their potential practical use as diagnostic or predictive tools. It is demonstrated that there are now three main lines of research in this area that do not appear likely to converge to a single model that encompasses all of the requirements for a useful tool with the attendant physical understanding of important nonlinear processes. The shortcomings of each method are identified, and a plan for improving this currently somewhat unpromising scenario is suggested.

Introduction

Recent work has illustrated the importance of a correct fluid dynamic formulation for the analysis of problems of oscillatory flow in rocket chambers. Most accepted analytical procedures are based on the notion that the principal gas motions are of an acoustic nature and hence irrotational. This idea is reinforced by the

Nomenclature

\[ a_0 \] Mean speed of sound
\[ \mathbf{e}_{r}, \mathbf{e}_{\theta}, \mathbf{e}_{z} \] Unit vectors in \( r, \theta, \) and \( z \) directions
\[ E_{m} \] Normalization constant for mode \( m \)
\[ k_{m} \] Wave number for axial mode \( m \)
\[ L \] Chamber length
\[ m \] Mode number
\[ M_{b} \] Mach number at burning surface
\[ n \] Outward pointing unit normal vector
\[ p \] Pressure
\[ P_{0} \] Mean chamber pressure
\[ r \] Radial position
\[ R \] Chamber radius
\[ S = k_{m} / M_{b} \] Strouhal Number
\[ t \] Time
\[ u' \] Oscillatory velocity vector amplitude
\[ U_r, U_z \] Mean flow velocity component
\[ W \] Function of \( r \) in axial velocity
\[ y \] Radial position \( (y = 1 - r) \)
\[ z \] Axial position
\[ \alpha \] Growth rate (dimensional, \( \text{sec}^{-1} \))
\[ \gamma \] Ratio of specific heats
\[ \delta \] Inverse square root of acoustic
\[ \epsilon \] Reynolds number \( \left( \delta \equiv \sqrt{\nu/a_0 R} \right) \)
\[ \zeta \] Vorticity function (defined in Eq. (16))
\[ \xi = (\delta k_{m})^2 / M_{b}^2 \] Viscous scaling parameter
\[ \lambda = M_{b} / k_{m} \] Inverse of Strouhal number
\[ \nu \] Kinematic viscosity \( (\nu = \mu / \rho) \)
\[ \rho \] Density
\[ \phi(r) \] Function defined in Eqs. (28-33)
\[ \chi(r,z) \] Vorticity spatial distribution function
\[ \psi(r) \] Exponential argument, Eq. (20)
\[ \omega' \] Amplitude of vorticity fluctuation
\[ \Omega \] Mean vorticity amplitude

Subscripts

\[ b \] Combustion zone
\[ m \] Mode

Superscripts

\[ * \] Dimensional quantity
\[ ' \] Amplitude of time-dependent part
\[ \cdot \] Vortical (rotational) part
\[ \cdot ' \] Acoustic (irrotational) part
\[ (r), (i) \] Real and imaginary parts
experimental fact that the measured frequencies of oscillation usually closely conform to expected acoustic mode frequencies. However, it has been demonstrated that rotational flow effects (vorticity) are of key importance in the combustion instability problem. In particular, it has been demonstrated that there exists a strong coupling between the acoustical and the vortical effects at the burning surfaces of the combustion chamber. These have major implications regarding the nonlinear modeling programs currently underway.

To recognize the key importance of the vortical corrections, note that the uncertainties in the earlier linear models related to the important linear flow turning effect have been clarified by introduction of a more realistic representation of the chamber gas dynamics. Not only does this result in much improved agreement between the linear theory and appropriate experimental situations, but a much enhanced physical understanding of the origins of flow turning is achieved. Similar improvement in understanding of the infamous "velocity coupling" concept has also been achieved (from the linear standpoint) although these insights have not yet been recognized by the combustion instability community.

The material presented in this paper is based on the strong belief held by the present authors that no nonlinear theory should ever be used that does not behave correctly in the linear limit (small amplitudes). To properly utilize this idea, it is clear that a fully validated linear model must first be available. This explains why there continues to be considerable emphasis on linear models in spite of the obvious need to understand the nonlinear problems of the real world of rocket development.

We will attempt to demonstrate that because of the newly introduced rotational flow corrections, a much improved linear model now exists. It follows that such a model represents a useful starting point in the construction of a valid nonlinear model.

In this paper we extend these successful linear concepts into the modeling of nonlinear combustion instability. It is clear that problems such as those involving high-amplitude limit cycle behavior with the accompanying mean pressure shift, triggering of linearly stable systems into high amplitude oscillations and so on must be brought to the same level of physical understanding already achieved in the linear case.

Three current approaches to modeling of nonlinear oscillation problems are described. Clearly, an indication of success of the theoretical work would be the convergence of the various approaches to some level of mutual agreement. Such a convergence does not appear on the current horizon. A plan for improving this situation is set forth in this paper.

The goal of the work described is to clearly set forth those features of nonlinear combustion instability that must be explained and made accessible to predictive analysis. It goes without saying that attention must also be directed to creating tools of use for interpreting experimental data.

There can be no confidence in existing or proposed predictive and interpretive analytical tools for combustion instability problems until certain basic features of the phenomenon are much more fully understood. Some of the truly basic aspects of the phenomenon are simply not addressed in any of the presently developing programs for nonlinear combustion instability analysis. A companion paper addresses the origin of what is often referred to in the literature as "irregular burning." This has come to be known as the DC shift, or the "mean pressure shift." The successful achievement of improve understanding of that important effect points the way to a more successful treatment of the general nonlinear instability problem.

Experimental data provide overwhelming evidence that the DC shift is the result of an enhanced propellant burn rate when gas oscillations parallel to the burning zone are present. Earlier models could not show a valid mechanism for the implied steady contribution to the heat transfer processes. Of necessity, the interactions must be of the second order in the wave amplitude. None of the obvious irrotational nonlinear effects in the governing equations can produce any important influence on the burning rate, since they are of the order of the wave amplitude, that is they are $O(\epsilon^2)$.

The goals of the work described in this paper are to:

- Review the present concepts for nonlinear modeling of combustion instability
- Identify the shortcomings of the existing models
- Present a plan for achieving a valid nonlinear model

**Experimental Evidence**

Availability of some new high-quality nonlinear experimental data collected in an international collaborative research program presents the opportunity to reexamine both the predictive capabilities of existing nonlinear models and the data interpretation methodology currently employed. These data sets contain much information that has not been fully explained, nor have they been subjected to searching scrutiny from the standpoint of nonlinear instability theory. In particular,
information of direct value in producing an improved nonlinear instability theory and data reduction procedure is evident in the published material.

The focus in this section is on important features of the data that were not addressed in analyses described in papers by Blomshied, et al., Mathes, and Harris. What was done in those studies is quite representative of current state of affairs in nonlinear combustion instability analysis. The perception is clearly that there are no reliable tools for nonlinear data reduction although this conclusion is usually not specifically mentioned in the reports. The quantitative analyses carried out in those studies were used mainly to test the validity of linear combustion instability prediction algorithms (SSPP), since much of the information gathered involved decaying low-amplitude pressure pulses.

The data were examined only qualitatively from the nonlinear point of view, since no readily applied quantitative theory has yet been established. The authors realize that these are controversial issues, but many years of observing the handling of nonlinear instability problems in the rocket industry supports the general view. The investigators hesitate to devote large amounts of time to any of the models, fully numerical or analytical, because there is simply no way to readily adapt them in the data analysis problem. These views are supported by the treatment of nonlinear effects demonstrated in the referenced reports.

In all of these programs only linear methods of analysis were employed for the quantitative handling of the test data by fitting linear theory to selected portions of the test data. The main objective of these programs, which was to improve understanding of nonlinear instability, was therefore not fully realized. A main goal of this section is to demonstrate that incorrect conclusions regarding both linear and nonlinear content of the data can be reached if a valid linear model is not employed. Again, there is no hope that a nonlinear understanding can be achieved if a linear one is not available at the outset.

Let us first examine the excellent data set produced by Blomshied and his coworkers. This program involved careful testing of twenty motors with a configuration typical of operational tactical rocket systems. Each test could be pulsed up to three times by pulses of preselected amplitude. A typical pressure amplitude vs. time trace is shown in Figure 1.

This particular plot (Motor Test No. 4, star forward geometry, 1000 psi mean pressure) illustrates several familiar nonlinear features that characterize the test series. The first pulse (decay) occurs at 0.97 seconds and has an amplitude of 43 psi. This pulse is clearly visible in Fig. 1. The second smaller pulse (25 psi) occurs at 1.96 seconds and leads to triggered growth to a high-amplitude limit cycle \( \epsilon = p'/c = 0.59 \) accompanied by a large DC shift which persists for about 1 second. The nonlinear oscillations then decay as the motor taildown begins. The behavior depicted in Fig. 1 was repeated in tests involving motors with full-length aft star, full-star, and cylindrical geometries. Both the initial (low-amplitude) and the triggered (high-amplitude) limit cycle amplitude was quite reproducible for tests with similar propellant and geometry.

This indicates that although nonlinear oscillations are clearly present, the test outcomes did not display chaotic behavior. In fact, there is much evidence of self-organization, which is a mode of behavior that is often encountered in nonlinear oscillators. Further discussion of this interpretation will be presented in later sections.

![Fig 1. - Pressure Trace, Motor No. 4](image)

The dominant frequency of both the low- and high-amplitude limit cycle oscillations corresponded closely to the first longitudinal acoustic mode. This observation is often used as the basis for theoretical models based on superposition of acoustic waves. Despite their low amplitude, the waveforms of the decaying oscillations produced in first pulse (pulse just before \( t = 1 \) in Fig. 1) show the same nonlinear attributes (steep wave front, etc.) present in the later triggered oscillation.

Figure 2 shows the waterfall data for a motor similar to that illustrated in Figure 1. In the case shown, the system was triggered by the second pulse, and the characteristic "steep wave" spectrum is produced. The peaks in the spectrum at higher frequencies are readily identifiable with transverse modes of oscillation. In general the behavior shown demonstrates the high amplitude limit cycle that appeared in several of the NAWC tests.
An important feature of the NAWC data set is the presence of a low-amplitude limit cycle oscillation in cases where triggered high-amplitude oscillations were not present. Figure 3 shows waterfall data for a motor of this type (Motor No. 5, Star-Aft, 1000 psi planned chamber pressure). In this motor only the first pulse was fired. The pulse occurred at \( t = 0.965 \) with an amplitude of 73 psi. This pulse amplitude was of sufficient strength to cause triggered instability had it been fired at a later point in the burn. Typically, pulses fired after mid-burn were likely to produce triggered response if of sufficiently high amplitude. There is little evidence of the decaying pulse in Fig. 3 because it has very nearly dissipated at the closest time-slice (t = 1.0 sec). The small second and third harmonic activity appearing at this time is probably related to the pulse.

The presence of a low-amplitude (less than 2 psi) oscillation at the first longitudinal acoustic frequency (approximately 320 Hz) is evident especially toward the end of the burn. Low-amplitude second and third harmonics accompany this feature.

There is also some evidence of electronic noise. 60-cycle oscillations and some of the harmonics (especially the 180- and 240-Hz harmonics) are clearly discernible. This seems to answer questions raised by other investigators as to the origin of the low frequency data. It was suggested that they arise from "electronic noise." However the waterfall plots for Motor No. 5 and similar plots for other motors in the test program make it clear that a real chamber gas oscillation was present even in the absence of pulsing. It is imperative to understand the significance of these oscillations in the context of the nonlinear behavior of the system.

The position taken by the NAWC investigators was that "...the oscillations observed are due to processes internal to the motor...although the spontaneous oscillations which occur naturally in this motor might be of concern in some real-world motor applications, they were not considered a subject worthy of intense study in this program." The present writers could not disagree more with this conclusion. Any spontaneous oscillations from whatever source are of the greatest possible significance because they indicate that the motor system is linearly unstable. In applying the SSPP in the data reduction, it is necessary that all system gains and losses are accounted for, even if they are related, as the report suggests, "...to naturally occurring acoustic oscillation due to turbulent fluid flow."

Obviously, no realistic nonlinear data interpretation can be undertaken without first achieving the most complete possible understanding of the motor operation from the linear standpoint. This is the reason that so much effort has been devoted to development of linear theories of combustion instability despite the fact that the most important practical problems involve nonlinear behavior. In addition to the classical linear effects such as pressure coupling, nozzle and particle damping, and so on, it is sometimes necessary to account for vortex shedding, turbulence, and distributed combustion effects in the linear modeling. There are also effects due to the creation of vorticity in the burning zone that are both the source of the flow turning effect and the origin of a new driving effect not represented in the SSPP. The significance of this information will be made clear as we further discuss the NAWC data.

Data reduction for decaying pulses was used to estimate linear growth rates despite the fact that the system appeared not to decay to a zero amplitude, but rather to a low-amplitude limit cycle. Plots like the one shown in Figure 4 were then drawn in which the predicted linear growth rates (from the Standard Stability Prediction Program, SSPP) were compared to those measured in the manner described. On this basis the authors concluded that there was "reasonable" agreement between the decay data and the linear stability prediction. If it were not for the evidence of the low-amplitude limit cycle, then this might be an acceptable conclusion. However, there are
two matters that require very careful consideration. The first is, of course, the fact the oscillations persist at a low level long after the pulse has decayed. The second is evidence that there are missing terms of great significance in the linear stability algorithm. Therefore no useful conclusion can be drawn from the apparent “acceptable” agreement between the linear theory and the decay data shown in Figure 4.

![Graph showing comparison between theoretical and experimental data.](attachment:image)

Fig 4 - Comparison to SSP Calculations

In a previous paper the present authors suggested that a simple nonlinear interpretative model based on the energy balance method be examined as a more acceptable data reduction scheme. This model is based on the simple idea that nonlinear losses of the second order in the wave amplitude are likely to dominate the motion in the case of the weak nonlinear decay discussed in reference to Figure 4. The basis for such an energy balance approach will be discussed in detail in the analysis section of the paper. The work by Beckstead and Jensen was a typical example of this approach. During this program, Beckstead found that a simple extension of the ideas of linear combustion instability theory seemed to result in more acceptable correlation with experimental data. His model was based on the addition of a nonlinear term to the usual linear model

\[ \frac{de}{dt} = \varepsilon \alpha \]  

Beckstead’s “pi correlation” extended the model to second order by the addition of a term proportional to the square of the oscillation amplitude:

\[ \frac{de}{dt} = \varepsilon \alpha + \beta e^2 \]  

This model formed the basis of extensive attempts to analyze experimental data by using more general formulas of this type in a curve fitting mode to arrive at the governing parameters such as \( \varepsilon \alpha \) and \( \beta \). Unfortunately, it was found to be impossible to find consistent results in this fashion, and the method was abandoned.

This failure motivated Culick to attempt a new nonlinear modeling effort based on an extension of the acoustic wave models used in linear combustion instability theory. The main working hypothesis was that since spectral analysis indicates that frequencies closely matching predicted acoustic natural modes dominate the data, then such modes represent an appropriate solution base. This synthesis of the classical mode shapes leads to a basic representation of the unsteady pressure field as

\[ \frac{p'}{p} = \sum_{n=1}^{\infty} \eta_n(t) \psi_n(r) \]  

where \( \psi_n(r) \) are the classical mode shapes and \( \eta_n(t) \) are time varying modal amplitudes. This is the Galerkin approach, and when Eq. (3) is substituted into the nonlinear wave equation and boundary conditions governing the chamber motion, a coupled set of oscillator equations results. Solutions of these equations then yield information on the system nonlinear behavior. Details are discussed in the excellent survey paper by Culick. This approach is the basis of the most widely accepted school of thought on nonlinear instability, and will be discussed in detail in the next section.

To finish the present survey of the NAWC data, let us examine the results of applying a model such as given in Eq. (2) but with the understanding that it cannot be used successfully in a blind “curve fitting” mode. What must be done to achieve a successful result is to utilize all of the data in a particular case that describes the situation. The key to success in the NAWC pulse decay data is to first estimate the limit amplitude to which the pulse decays. If this information is used together with the data points describing the decay, an improved understanding emerges. Figure 5 shows a typical outcome of such a calculation. The results indicate that if a linear model is chosen (as it was in producing the data for Figure 4) a negative linear growth is naturally required. If, on the other hand, a second order nonlinear model is used (Eq. (2) along with the system limit amplitude averaged from the values present just before and just after the pulse decay) the solid curve shown in Figure 5 is the outcome. It not only represents a much better fit to the data points, but it more nearly portrays the actual system behavior. Of greatest significance is that it predicts that the system is linearly unstable. This outcome should be quite obvious anyway since the system oscillated at a low amplitude limit cycle both before and after the pulse. We must therefore determine the reasons for the apparent failure.
of the linear stability code. An obvious difficulty is that the system is behaving in nonlinear fashion even though the
amplitude of oscillation is not particularly large. In order
for the linear stability assessment to be working correctly, it
must necessarily indicate that the system is linearly unstable.
Clearly there is something missing in the linear code. We
will demonstrate in the next section that at least part of
the missing information is a term in the stability energy balance
arising from production of vorticity at the burning
surfaces.

This could not be done in the original formulation, since
the flow turning model was based on a one-dimensional
analysis, which precluded a sufficiently detailed physical
understanding. It was shown by Flandro\textsuperscript{1-3} that the true
origin of flow turning is in the interaction between vorticity and the mean flow field in the stability integral
\begin{equation}
\alpha_{FT} = \frac{M_b}{2k_mE_m^2} \frac{a_0}{R} \int \int \int (U \times \omega') \cdot \nabla p_m' \, dV
\end{equation}

where
\begin{equation}
E_m^2 = \int \int (\rho_m')^2 \, dV
\end{equation}

and the importance of the vorticity fluctuation is clearly
indicated. Earlier combustion instability modeling was
based on the idea that the vorticity is small. That is, of
the order of the mean Mach number, \(M_b\). However, it
has now been proved in a variety of ways that this
assumption is incorrect. The unsteady vorticity is instead
proportional to
\begin{equation}
\omega' = \frac{1}{M_b} - S
\end{equation}

where \(S\) is the Strouhal number. Several critics have
indicated that this must indicate an error in the calculation,
since the gas motion must reduce to the classical acoustic
boundary layer result in the limit as \(M_b \to 0\), but (5)
seems to suggest that the vorticity "blows up" as the
Mach number approaches zero. Careful reduction of the
solution for the vorticity solution with blowing to the
limiting case as the Mach number goes to zero verifies
that the solution indeed reduces to the classical unsteady
boundary layer solution. The proof was worked out by
the present authors, and can also be found in the analysis
by Majdalani.\textsuperscript{5}

Equation (4) has been criticized as not resembling
the original flow turning correction. For example, it
appears as a volume integral instead of an integral over
the chamber surface. Of particular concern is that it
appears difficult to determine the distribution of the
unsteady vorticity throughout the chamber in order to
perform the integration indication in Eq. (4). However,
the reduction to a general surface integral has already
been demonstrated in earlier papers. In fact it is readily
shown that
\begin{equation}
\alpha_{FT} = \frac{-M_b}{2E_m^2 k_m^2} \frac{a_0}{R} \int \int \int (\nabla p_m' \cdot \nabla p_m') \, dS_b = \frac{1}{2E_m^2 k_m^2} \frac{a_0}{R} \int \int \int (p_m' \cdot \nabla p_m') \, dS_b
\end{equation}

This result is completely general. It holds for any
chamber geometry. It also shows that the only
information needed to calculate the flow turning loss is

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Application of Simple Nonlinear Theory to NAWC Pulse Decay Data}
\end{figure}

Missing Parts of the Linear Stability Model

It is now necessary to demonstrate that introduction
of an improved linear stability model explains
the apparent inconsistency between the results shown in
Figures 4 and 5. The former indicates for all the pulse
decay data that the system is necessarily stable and decays
to zero amplitude in linear fashion. The apparent growth
rates compare fairly well with predictions made with the
current version of the SSPP. However, including other
elements of the data such as the presence of the low-
amplitude limit cycle, another picture appears. As Figure
5 shows, a much better representation of the system is
found if it is treated as a nonlinear system. However,
that leaves us with the need to explain why the system is
linearly unstable. The answer can be found in the linear
vorticity corrections.

Several recent studies show the great importance of
allowing the gas motions to be rotational.\textsuperscript{1-5} In fact, this
is the only way to satisfy the no slip condition at the
propellant surface. Reference 1 shows that incorporation
of vorticity leads to a full physical understanding of the
flow turning term, which is one of the dominant terms
in the SSPP. It also allows for a nonambiguous
calculation of the flow turning loss for a three-
dimensional combustion chamber of arbitrary geometry.
the magnitude of the acoustic pressure gradient at the burning surface. This information is available within the standard stability calculation as part of the acoustic mode calculations required for each part of the loss/gain balance.

In addition to the flow turning, the inclusion of the rotational flow corrections shows that it is necessary that there be a new stability integral

$$\alpha_{\text{VORT}} = -\frac{1}{2E_m} \left( \frac{a_0}{R} \right) \oint n \cdot \hat{\nabla}_{m} dS = \frac{M_b}{E_m} \left( \frac{a_0}{R} \right) \oint \left( \nabla'_{m} \right)^2 dS$$

(8)

where the subscript “vort” is used here to denote the additional correction required by the satisfaction of the continuity equation at the propellant surface. Full discussions of this important term are presented in the recent literature.1-5 Notice that this surface integral can be readily evaluated for any combustion chamber geometry. The only information required is the local value of the pressure fluctuation amplitude, again, this is a quantity already required for other key stability elements such as the pressure coupling. In fact, Eq. (8) closely resembles the latter effect and is therefore an important contribution to the system linear energy balance.

It is useful to finish the section by demonstrating that the new view of linear stability better agrees with experimental data. A recent test motor, denoted “Test Case No. 1” experienced unexpected combustion instability in a series of firings conducted by Blomshiel.12 The standard stability code predicted that this motor configuration should be stable in all modes. However, the motor oscillated in both first and second longitudinal mode during the first part of the motor burn. Figures 6 and 7 shows the effect of the vortical flow corrections on the SSPP prediction for the first mode.

Figure 7 compares the net first mode growth rate from the SSPP with and without the vortical correction term from Eq. (8).

Other verification of the corrected unsteady gas motion are presented in some current literature and will not be repeated here. There is very considerable evidence that the rotational flow effects must be accounted for in combustion stability calculations. Although the full details have not yet been worked out for the NAWC data discussed earlier, there is good qualitative evidence that incorporation of the new corrections will lead to better agreement between experimental and analytical stability determination. Examination of the NAWC data shows that inclusion of the correction of Eq. (8) will yield a linearly unstable prediction for many of the motor test configurations. This, in turn, makes it much easier to interpret the nonlinear features of the data.

For emphasis, we have demonstrated the importance of a valid linear stability model as the foundation for a nonlinear model. We must reject nonlinear models that do not reduce to the correct linear behavior for small amplitudes. This is not to say that there are no nonlinear gain/loss effects that only appear in a nonlinear treatment. On the contrary, some very potent mechanisms must be accounted for if a valid system model is to be constructed. The attributes of such a model are the subject of the next section.

Types of Nonlinear System Models

There are presently three distinct approaches to the modeling of nonlinear combustion instability. The following is an attempt to describe both the strengths and the weaknesses of these models with the objective of identifying a productive path for future efforts. Although some elements of the discussion may seem harsh, there is certainly no intention to impugn any particular school.
of thought. The almost overwhelming complexity of the problem seems to justify the exploration of almost any set of simplifying assumptions.

The basic features that must be incorporated into any successful nonlinear model are: 1) reduction to correct linear behavior for sufficiently small amplitude, 2) accommodation of loss mechanisms related to energy dissipation in regions of nonuniformity such as shocklike waves, 3) correct fluid mechanics allowing for the influence of strongly coupled acoustical/rotational oscillations, and 4) a sufficiently detailed nonlinear model for the flame zone. If the latter element is not addressed, then there will never be a way to incorporate the elaborate data and combustion models currently being produced by those studying nonlinear instability from the standpoint of the basic chemistry of the propellant decomposition and combustion. It goes without saying that no model is of much use if it cannot readily account for all of the known attributes of nonlinear system behavior. These include the approach to a limit cycle, the possibility of triggering causing the system to undergo transition from one limit cycle to another (with either higher or lower amplitude), and the DC shift phenomenon.

We now attempt to assess the existing nonlinear modeling approaches in terms of the satisfaction of the requirements just identified. The three current approaches can be classified as:

I. Superposition of a set of nonlinear coupled oscillators with the main object of accounting for transfer of energy between the modes.

II. Full numerical treatment of the governing equations with boundary conditions based on ad hoc models of the nonlinear combustion effects.

III. Nonlinear extension of the classical system energy balance approach with the main objective of simplifying the incorporation of wave steepening and shock loss effects.

The second method, II, would appear at first glance to offer the most hope for successfully modeling the nonlinear system behavior, since it is based entirely on solution of the fundamental conservation equation. Eventually this may well be true. However, experience shows that application of a numerical model of this sort does not eliminate the need to provide analytical models for key elements of the problem. In particular, handling of the nonlinear behavior of the burning zone is an important difficulty. Simply introducing an “educated guess” as to which type of terms in the equations are most likely to lead to the type of nonlinear behavior observed in experiments appears to be a dangerous approach. This is not to say that such calculations cannot be used productively in the “numerical experiment” mode. Some very important current problems with the numerical approach involve the difficult influence of numerical diffusion, resolving subtle interactions, especially those near the burning zone, overspecifying the boundary conditions because valid models for the physical processes of importance do not automatically appear, and so on. It seems safe to say that, eventually, this will be the preferred way to approach nonlinear instability. The ability to handle any geometrical configuration and grain features is a major benefit. However, there is just too much analytical knowledge missing from the problem to accomplish this now.

Approach I has received the most attention in recent times. It is appealing from the standpoint that it proceeds from a well-established base for which there is an extensive literature on the treatment of nonlinear system behavior. The difficulty that we see with the present implementation of this method is that it is too strongly focused on one mechanism of nonlinear gas interaction, namely the flux of energy between the components of the system of oscillators. It is not possible to directly include, for example, the energy loss in discontinuities. This is because the structure itself is a representation of the transition or steepening of the system into a shocklike form. A major difficulty is that the only nonlinear mechanism that naturally arises in this model is mode coupling. All other nonlinear effects require additional modeling. For instance, the current model cannot be used to recover the correct linear behavior in the limit of small oscillations. The only recourse is the familiar “patching” approach in which additional nonlinear interactions arising from other models are incorporated. Recent works attempt to include effects of vorticity as “noise.” This disregards the coupling at the boundaries between the acoustic and vortical modes. In other words, the coupled oscillator approach does not represent a natural way to treat coupled acoustical/vortical waves. There can be no doubt that the correct inclusion of the coupled vortical gas motions in a rocket greatly improves the correspondence of models with the experimental data.

This leads directly to a new approach which is based on the application of energy methods. The inherent benefits of the latter are obvious. However, in view of our findings regarding the importance of rotational flow in the linear problem, it is necessary that such effects be included at the outset in the nonlinear problem. Such a model is formulated in the following section. As with the other approaches, Method III has its own set of limitations. The one of usual concern is that it does not immediately allow for mode coupling since it is based
on the idea that the system behavior can be described in terms of a global amplitude effect. The key idea is that in nonlinear combustion instability, the steepening into a shock-like system occurs rapidly on the scale of the transitions between limit cycles. Then the net flux of energy between the modal components vanishes; the wave is fully steepened, and the transitions are controlled by the potent nonsentropic energy loss in the steep wavefront. All attributes of nonlinear behavior appear in models of this type. Furthermore, there is considerable experimental data that verifies this view; that is, the relative amplitudes of the acoustic modal components stays very nearly fixed throughout triggering or decay between limit cycles.

An equally potent mechanism for controlling nonlinear behavior can be found in the acoustical/vortical interactions, and in particular the nonlinear energy dissipation in large amplitude vortical shear waves. Neither of these two mechanisms arises naturally in method I. Therefore, we set forth in the next section to formulate a new model which incorporates all of the most important energy transfer processes in a natural way.

Nonlinear Energy Balance Formulation

The primary goal at this stage, is to introduce in a clear and concise manner, the fundamental concepts which underlie our new approach to nonlinear instability. Dimensionless variables are not introduced initially, nor has the flowfield been separated into time averaged and oscillating parts as in the traditional instability modeling approach. We begin our analysis with a brief derivation of the classical acoustic energy balance model.

The equations describing the conservation of mass and momentum for a fluid system are respectively,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot m = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) + \omega \times \mathbf{u} - T \nabla s = -\nabla h - \frac{\mu}{\rho} \left( \nabla \times \omega - \left( 2 + \frac{\lambda}{\mu} \right) \nabla (\nabla \cdot \mathbf{u}) \right)$$

(9)

The thermodynamic relationship,

$$T \nabla s = \nabla h - \frac{1}{\rho} \nabla p$$

(10)

has been employed. The interesting physical mechanisms inherent within the momentum equation are then suppressed with the introduction of the three classical "i" constraints: irrotational, inviscid, isentropic.

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla h_0 = -\sqrt{\frac{\gamma}{\gamma - 1} \rho + \frac{1}{2} |\mathbf{u}|^2}$$

(11)

The simplified field equations are subsequently combined in the following fashion,

$$h_0 \frac{\partial \rho}{\partial t} + m \cdot \frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot (h_0 \mathbf{m})$$

(12)

Since the flow is assumed to be isentropic, pressure and density are intimately related,

$$d \rho = \frac{\partial p}{\partial \rho} \left|_s \right. dp + \frac{\partial p}{\partial s} \left|_s \right. ds = c^2 dp$$

(13)

($c$ is the sound speed). Use of the preceding statement of isentropy, along with the thermodynamic relationship,

$$\rho e = \frac{p}{\gamma - 1}$$

(14)

($e$ is the specific internal energy), valid for an ideal gas, allows one to reduce the acoustic energy balance, after integration over the flowfield domain and subsequent application of the Reynolds transfer theorem,

$$\frac{D}{Dt} \int \nabla f dV = \frac{\partial}{\partial t} \int \nabla f dV + \int \mathbf{F} \cdot \nabla dS$$

(15)

to a very simple and elegant form.

$$\frac{D}{Dt} \int \left( e + \frac{1}{2} |\mathbf{u}|^2 \right) dV = - \int \mathbf{F} \cdot \nabla dS$$

(16)

The observant reader undoubtedly recognizes the preceding equation as nothing more than an abridged version of the first law of thermodynamics, which when restored to its full glory, assumes the more ominous form,

$$\frac{D}{Dt} \int \rho \left( e + \frac{1}{2} |\mathbf{u}|^2 \right) dV = \int \mathbf{F} \cdot \mathbf{u} dV + \int \left( t \cdot \mathbf{u} - \mathbf{q} \cdot \mathbf{n} \right) dS$$

(17)

where $\mathbf{f}$ is the external body force per unit mass, $t$ is the traction or stress, and $\mathbf{q}$ is the heat flux. The equation essentially states that the rate of change of the total energy of a material volume is equal to the rate at which work is being done on the volume plus the rate at which heat is conducted into the volume.

A reasonable conjecture then that would be that a more appropriate starting point for devising a generalized energy balance formulation would be the first law of thermodynamics itself. The classic difficulty with the first law of thermodynamics, however, is that it does not differentiate between recoverable and non-recoverable internal energy.
A more appropriate starting point then would be to derive an equation for,
\[ \frac{D}{Dt} \int \rho \left( \epsilon_{\text{reversible}} + \frac{1}{2} |\mathbf{u}|^2 \right) dV = ... \] (18)

A wave, in the classical sense, is the interaction between two forms of energy. In the absence then of any dissipative or growth mechanisms, this interchange proceeds *ad infinitum*.

Essentially then we want to model the growth of a coherent acoustic/rotational wave system. The rotational component of the wave system is necessary so that physically correct boundary conditions are satisfied. What is important to realize, is that the rotational component of the wave system oscillates temporally at the same frequencies as the modes of the chamber...and in other words...when measuring the pressure using a transducer we would not be able to directly determine that their was in fact a rotational component to the wave system. *Vorticity cannot be incorporated by patching it on as an afterthought.*

The first step then is to derive an equation for
\[ \frac{D}{Dt} \int \rho e_{\text{reversible}} dV = ... \] (19)

Recall the familiar thermodynamic relationship,
\[ \frac{D}{Dt} \rho \mathbf{v} + \rho \mathbf{v} \cdot \mathbf{u} = \rho T \frac{D}{Dt} \mathbf{s} \] \quad (20)

We need to appeal second law of thermodynamics, as formalized by the Clausius-Duhem inequality,
\[ \frac{D}{Dt} \int \rho s dV \geq - \int \frac{\mathbf{q} \cdot \mathbf{n}}{T} dS \] \quad (21)

For a reversible process then,
\[ \rho \frac{D}{Dt} s_{\text{reversible}} = - \nabla \cdot \left( \frac{\mathbf{q}}{T} \right) \] \quad (22)

After introducing Fourier's law of heat conduction, \( \mathbf{q} = -k \nabla T \), we obtain the following result
\[ \rho \frac{D}{Dt} e_{\text{reversible}} + \rho \mathbf{v} \cdot \mathbf{u} = \rho T \frac{D}{Dt} s_{\text{reversible}} = \] \quad (23)

\[ = \nabla \cdot (k \nabla T) - k \frac{[\nabla T]^2}{T} \]

In the absence of any heat transfer effects, or if they are confined to a very thin region, as within a shock wave, the classic isentropic relationship between the thermodynamic variables is recovered. The next step is to derive an equation for
\[ \frac{D}{Dt} \int \frac{1}{2} \rho |\mathbf{u}|^2 dV = ... \] (24)

Projecting the momentum equation,
\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \nabla |\mathbf{u}|^2 \right) = - \nabla p - \mu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) - \left( 2 + \frac{\lambda}{\mu} \right) \nabla \cdot \left( \mathbf{u} (\nabla \cdot \mathbf{u}) \right) \] \quad (25)

along the direction of the velocity vector, yields an equation for the rate of change of kinetic energy. (In the more general formulation, when the flow field variables are separated into time average and oscillating components, our interest would be in the rate of change of the unsteady kinetic energy).
\[ \rho \frac{D}{Dt} \left( \frac{1}{2} |\mathbf{u}|^2 \right) = - \mathbf{u} \cdot \nabla p - \mu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \cdot \nabla \cdot \left( \mathbf{u} (\nabla \cdot \mathbf{u}) \right) \] \quad (26)

With the introduction of a few basic vector identities, this can be rewritten as,
\[ \rho \frac{D}{Dt} \left( \frac{1}{2} |\mathbf{u}|^2 \right) = p \nabla \cdot \mathbf{u} - \mu \left( \nabla |\mathbf{u}|^2 + (2 + \frac{\lambda}{\mu}) \mathbf{u} \cdot \nabla |\mathbf{u}| \right) \] \quad (27)

The equations for the rate of change of kinetic energy and the rate of change of reversible internal energy are combined by using the term describing the expansion of work, \( p \mathbf{v} \cdot \mathbf{u} \) Integration over the flowfield domain then yields the following generalization of the classical acoustic energy balance,
\[ \frac{D}{Dt} \int \left( \epsilon_{\text{reversible}} + \frac{1}{2} |\mathbf{u}|^2 \right) dV = \] \quad (28)

\[ - \int \left( \mu |\mathbf{u}|^2 + (\lambda + 2 \mu) \nabla \cdot \mathbf{u} + k \frac{[\nabla T]^2}{T} \right) dV - \int (p - (\lambda + 2 \mu) \nabla \cdot \mathbf{u}) \mathbf{n} + q \cdot \mathbf{n}) dS \]

(Note: the term \( \mu \nabla \cdot (\omega \times \mathbf{u}) \) vanished since \( \mathbf{n} \) is parallel to \( \mathbf{u} \) at the boundaries.)

The various terms on the right of Eq. (28) can be readily interpreted in terms of their effect on the nonlinear system energy balance. The first term (representing the direct loss of system energy when vorticity is produced in the flame zone interactions) comes from the energy dissipation function. This is a self-limiting effect that has not been accounted for in any of the existing models. It is important because the unsteady vorticity amplitude is of the order of the inverse of the mean flow Mach number as shown in Eq. (6). This term represents a very important sink of
system energy. The second term is the origin of the shock wave energy loss. Notice that it plays a major role whenever there are steep temperature gradients as in a discontinuity. The last term represents several effects including flow work, heat transfer and viscous dissipation at the chamber boundaries. The most important part of the flow work term in the nonlinear sense will come from coupling terms in the burning zone. These effects are presently under intense study and will be described in a later paper.

Conclusions

The vital importance of inclusion of rotational flow effects in the nonlinear combustion instability problem has been demonstrated. Major improvements in agreement with experimental data are found in this way. In constructing nonlinear analysis tools, it is crucial that the models reduce to the linear behavior when the wave amplitude decreases to a small value. Existing models do not exhibit this important attribute. Efforts to patch on the effects of vorticity and entropy production as “noise” will not yield the expected results; such an approach does not account for the strong vortical/acoustical coupling in the flame zone.

The extended energy balance method shows considerable promise as the foundation for a powerful tool for nonlinear instability analysis. Effects of shock wave losses and acoustic/vorticity interactions are accounted for directly in a straightforward fashion. All known attributes of nonlinear behavior are accommodated in a simple set of readily understood physical mechanisms. Additional work on the nonlinear combustion zone model must be completed to yield the desired predictive/interpretive capability.

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